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Isospectrality of Quasinormal Mode Frequencies in Four- and Five-Dimensional Black Holes

Perturbation theory on Schwarzschild and Black String spacetimes

Thesis MSc. Particle- & Astrophysics

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Abstract

Metric perturbation theory serves as a powerful tool for exploring the mathematical properties of black holes. In this work, we present a comprehensive and notationally consistent review of the formalism and apply it to the Schwarzschild black hole, both in a covariant framework and within a specific coordinate system. To handle the most computationally demanding steps, we develop a dedicated script in Mathematica. Within this formalism, we derive the Regge-Wheeler and Zerilli equations and explicitly demonstrate that their potentials are connected through a Darboux (or Chandrasekhar) transformation. The existence of such a transformation demonstrates that the equations are isospectral, meaning that the potentials have the same spectrum of quasinormal-mode frequencies. We extend the Schwarzschild metric with one extra spatial dimension to construct the five-dimensional metric of a black string. Following the methodology used in four dimensions, we examine the presence of isospectrality in five dimensions. Our analysis reveals that the odd-parity perturbation equations can be decoupled into two independent equations. However, due to the large number of equations and variables, we are unable to fully decouple the even-parity perturbation equations. As a result, it remains an open question whether the black string spacetime exhibits isospectrality.

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1 Introduction

In our everyday experience, we perceive the universe as consisting of four dimensions: three describing space and one representing time, which together form the fabric of spacetime. Spacetime can be curved, creating the effect we recognize as gravity, a phenomenon elegantly described by Einstein's General Theory of Relativity (GR). Despite its extraordinary accuracy, GR cannot currently be reconciled with quantum theory. The search for a unifying theory, a theory of Quantum Gravity, has become a major focus of research and is often referred to as the "holy grail" of theoretical physics. One promising approach to merging GR and quantum theory involves rethinking our understanding of gravity itself. Over recent decades, researchers have proposed various (sometimes unconventional) solutions to this unification puzzle, including theories suggesting that gravity may operate in higher dimensions.

One of the earliest higher-dimensional theories of gravity was Kaluza-Klein theory, which aimed to unify gravity and electromagnetism within a single framework. It proposed the existence of an additional spatial dimension, bringing the total number of spacetime dimensions to five. This extra dimension, however, would be curled up into a tiny circular shape, making it nearly impossible to detect. In this model, the additional dimension manifests itself as an electromagnetic field [1, 2].

Kaluza-Klein theory laid the foundation for the development of (Super-)String Theory (ST). ST extends the ideas of Kaluza-Klein theory by suggesting that spacetime has even more dimensions. To maintain mathematical consistency, ST requires a ten-dimensional universe, consisting of nine spatial dimensions and one temporal dimension. Similar to Kaluza-Klein theory, these extra dimensions are compactified into minuscule, circular strings. The inclusion of additional spatial dimensions enables gravity to be incorporated alongside the other three fundamental forces (that is, the electromagnetic, weak, and strong forces), and in fact, ST *requires* GR in order to be consistent [3]. In these models, gravity can propagate through the extra dimensions that the other forces do not interact with or do so much more weakly. This could potentially explain why gravity is significantly weaker than the other fundamental forces and provides a framework for unification at higher energy scales.

While ST has made significant strides toward formulating a theory of Quantum Gravity, it has some severe shortcomings; it remains unverified experimentally and faces major mathematical and conceptual hurdles, such as reproducing the Standard Model of particle physics. The main experimental challenge arises from the fact that the six extra spatial dimensions form a compact space on the order of the Planck length, making them them far too small to be detected with current technology [3]. To determine whether higher dimensions exist, we must therefore investigate their effect on macroscopic objects, such as black holes.

If higher dimensions exist, black holes would also extend into these dimensions and could exhibit properties that differ from their four-dimensional versions [4]. Some of these unique features might even be observable in high-energy experiments through interactions with Standard Model fields [5–8]. Studying the properties of higher-dimensional black holes could thus offer valuable clues about the validity of ST as a higher-dimensional theory of gravity.

One approach to studying the properties of black holes is through perturbation theory. A perturbation refers to a small, sudden deformation of a black hole's shape, which can occur when two black holes merge or when matter falls into it. As the black hole settles back into equilibrium, it emits gravitational waves with specific frequencies that gradually decay over time. These characteristic oscillations, known as quasi-normal modes (QNMs), provide a unique signature of the perturbed black hole and can be observed through gravitational wave detections [9].

Several well-established formalisms exist for describing black hole perturbations, each offering distinct advantages depending on the specific application, such as stability analysis, gravitational wave emission, or the study of binary systems. There are numerous extensions of these formalisms accounting for certain properties of the black hole spacetime, including rotation [10], charge [11, 12], and dimensionality [13–15]. For the fundamental case of a four-dimensional Schwarzschild black hole, pioneering contributions were made by Regge and Wheeler [16], Vishveshwara [17], Zerilli [18] and Chandrasekhar [19, 20]. Later, Martel and Poisson [21] developed a comprehensive formalism that is covariant and gauge-invariant, integrating and refining key results from earlier work. This thesis will employ their formalism.

The standard method for analyzing metric perturbations in Schwarzschild spacetime involves decomposing the metric into two-dimensional submetrics: a Lorentzian manifold and a spherically symmetric subspace [9]. Perturbations are introduced via a perturbing metric, which is then expanded using spherical harmonics. This allows for the separation of the angular component and the classification of perturbations into two distinct parities: even and odd. By solving the perturbed (vacuum) Einstein equations in these decompositions, one finds the well-known result that they reduce to two independent wave-like equations with a potential, each corresponding to a specific parity. The *Zerilli equation* governs even-parity perturbations, while the *Regge-Wheeler (RW) equation* describes odd-parity perturbations. At first sight, these equations appear different and unrelated. Interestingly though, it can be shown that they are in fact related and therefore share an identical spectrum of QNM frequencies, a phenomenon known as *isospectrality*. This property was first identified by Chandrasekhar and Detweiler [22]. Isospectrality in a given spacetime can be established by demonstrating the existence of a mathematical transformation between even- and odd-parity perturbations, known as the Chandrasekhar transformation. This was recently demonstrated to be a special case of the more general Darboux transformation by Glampedakis et al. [23].

While isospectrality holds for classical black hole spacetimes such as Schwarzschild, Reissner-Nordström and Kerr, this generally not true for black holes in alternative theories of gravity [24]. Several studies have reported the breaking of isospectrality in such alternative theories, with recent investigations presented in Ref. [25–28]. However, this does not imply that extensions of GR inevitably break isospectrality; a recent study by Cano and David [29] indicates that certain extensions of GR do, in fact, preserve it. If isospectrality is broken, this could lead to observable effects in gravitational wave data (for details on how this manifests, see Ref. [30, 31]). As a result, detecting such deviations could provide valuable insights into the possible existence of alternative theories of gravity.

To assess whether ST is a viable higher-dimensional theory of gravity, we can examine whether isospectrality holds in ten-dimensional black holes. However, starting at such a high dimensionality is clearly impractical. A more feasible approach is to first study lower-dimensional cases, such as five-dimensional ST, where black holes are known as *black strings*. his thesis will therefore investigate isospectrality within the black string spacetime.

The first section of this thesis provides a comprehensive and self-contained review of the theory of metric perturbations in Schwarzschild spacetime, formulated in a way that allows for extension to higher dimensions. Using the formalism developed by Martel and Poisson [21], we derive the odd- and even-parity perturbation equations, reduce them to the Regge-Wheeler and Zerilli equations, and demonstrate their isospectrality.

In the second section, we apply this formalism to the black string spacetime, following the same steps to determine whether a similar transformation exists between odd- and even-parity perturbations. The calculations presented in this thesis are partially carried out using three custom-developed Mathematica scripts: 4d_pert_covariant, 4d_pert_coordinates and 5d_pert_coordinates [32].

2 Metric Perturbations of the Schwarzschild Spacetime

As mentioned in the introduction, there are several approaches to studying perturbations of black hole spacetimes. In this section, we adopt the formalism of metric perturbations developed by Martel and Poisson [9, 21]. This choice is motivated by its relatively simple and well-structured extension to higher dimensions (at least in principle). Such an extension is essential, as working in five dimensions introduces a greater number of metric components and corresponding equations. While other widely used formalisms, such as the Newman-Penrose (or Geroch-Held-Penrose) approach, have been extended to higher dimensions in recent studies [15, 33–35], they are not particularly user-friendly, making the Martel-Poisson formalism a more practical choice for our analysis.

In this section, we derive the RW and Zerilli equations for the Schwarzschild metric and show that the QNM frequencies in this spacetime are isospectral, carefully outlining the various steps involved in the process. The goal of this comprehensive review is primarily to combine and present in a self-consistent manner the information scattered throughout the literature, as intermediate calculations are seldom included due to their complexity, despite the fact that they often contain non-trivial or subtle details. Once this detailed analysis of the four-dimensional case is complete, we extend the procedure to the higher-dimensional black string spacetime, following a similar approach, in Section 3.

The process of applying the formalism of metric perturbations to the Schwarzschild spacetime is outlined in Figure 1, reflecting the structure of this thesis. In the following section, we will show how to decompose a four-dimensional metric into two-dimensional submetrics, separating the (t, r)- and (θ, ϕ)-coordinates. Section 2.2 focuses on applying a linear perturbation to the Schwarzschild metric $g_{\mu\nu}$ in the form of a perturbing metric $\gamma_{\mu\nu}$. We make explicit in Section 2.3 the covariant derivatives of each metric, discuss the commutation relations, and provide some useful identities that will be important for the subsequent analysis. Then, in Section 2.4 we derive the linearized curvature quantities, such as the Christoffel symbols, Ricci tensor, and Einstein tensor. By setting each component of the linearized Ricci tensor to zero, we obtain a system of linearized vacuum Einstein equations. We specifically need the Einstein tensors for the even-parity sector, as will be explained in Section 2.8.2. In Section 2.5, we expand the perturbing metric $\gamma_{\mu\nu}$ in terms of spherical harmonics, separating it into an odd- and even-parity sector. Section 2.6 will discuss the gauge freedom for each parity and demonstrate how applying the Regge-Wheeler gauge can eliminate some of them. We then insert the gauge-fixed expressions for the perturbing metric—expanded into spherical harmonics and separated by parity—into the vacuum Einstein equations. The resulting system of coupled PDEs can, remarkably, be decoupled, yielding the RW equation for the odd-parity sector (Section 2.7) and the Zerilli equation for the even-parity sector (Section 2.8).

We will discuss two methods for decoupling each system: (1) by inserting the Schwarzschild coordinates into the covariant vacuum Einstein equations and solving the system as a set of coupled partial differential equations (PDEs), or (2) by rewriting the system in its covariant form and introducing the covariant RW and Zerilli functions (a method developed in [9]). Both approaches will be worked out in detail for completeness.

Terminology and notation in the literature are frequently a source of confusion. While many authors use the terms *axial* or *magnetic* to refer to odd-parity perturbations, and *polar* or *electric* for even-parity ones, we will avoid using these terms. Additionally, we have chosen a slightly different notation from the Martel-Poisson formalism to minimize potential confusion with overlapping symbols. A detailed correspondence between this thesis and their work is provided in Appendix A. Throughout this thesis we will adopt natural units ($c = G_N = 1$) and use the mostly-plus convention for the Minkowski metric $(- + + +)$.

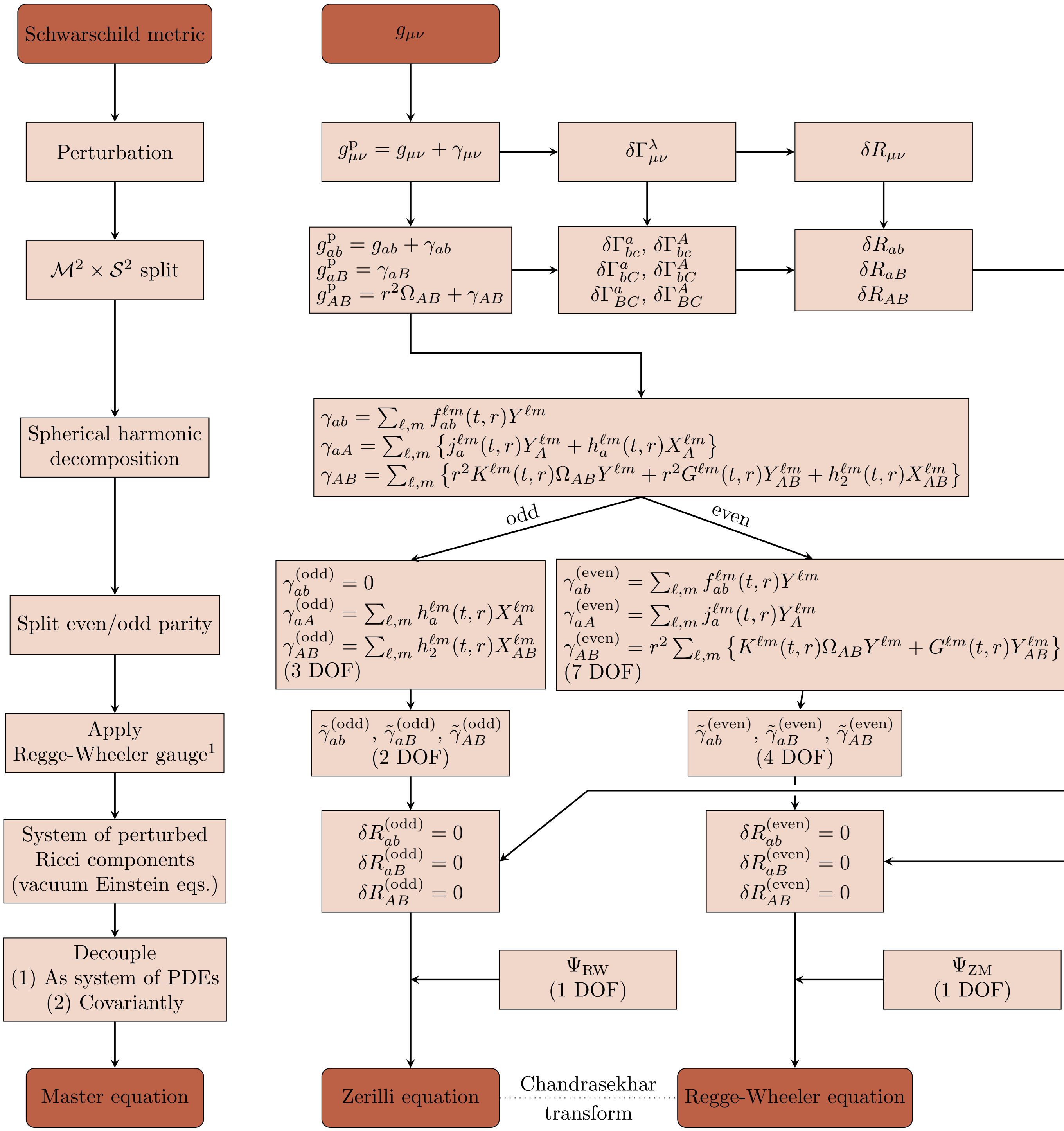

Figure 1: Flowchart visualizing the steps involved in the formalism of metric perturbations of the Schwarzschild spacetime. The number of degrees of freedom (DOF) are indicated in the last steps.

2.1 $\mathcal{M}^2 \times \mathcal{S}^2$ decomposition

Due to the spherical symmetry of the Schwarzschild spacetime, it is possible to decompose the metric into radial and angular components, allowing us to treat the perturbations of each part separately. In this context, the line element can be expressed as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) := g_{ab}dx^a dx^b + r^2\Omega_{AB}dx^A dx^B, \tag{1}$$

where

$$f(r) := 1 - \frac{2M}{r} \tag{2}$$

and M the mass of the black hole. In this decomposition, the four-dimensional manifold $\mathcal{M}^4$ will be represented as a product of two-dimensional submanifolds, that is, $\mathcal{M}^4 = \mathcal{M}^2 \times \mathcal{S}^2$. Here $\mathcal{M}^2$ refers to the submanifold spanned by the t and r coordinates, and $\mathcal{S}^2$ is the submanifold of the two-sphere spanned by the angular coordinates θ and ϕ. The lowercase Latin indices are used to represent t and r,

$$x^a = (t, r), \qquad a = 0, 1, \tag{3}$$

while the capital Latin indices run over the angular coordinates,

$$x^A = (\theta, \phi), \qquad A = 2, 3. \tag{4}$$

The four-dimensional metric $g_{\mu\nu}$ has coordinates indicated by Greek indices, i.e. $x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, and its components are given by

$$g_{\mu\nu} = \begin{bmatrix} g_{ab} & 0 \\ 0 & g_{AB} \end{bmatrix} = \begin{bmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{bmatrix}, \qquad g^{\mu\nu} = \begin{bmatrix} g^{ab} & 0 \\ 0 & g^{AB} \end{bmatrix} = \begin{bmatrix} g^{ab} & 0 \\ 0 & \frac{1}{r^2}\Omega^{AB} \end{bmatrix}. \tag{5}$$

Here Ω_{AB} represents the metric on the unit two-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 = \Omega_{AB}dx^A dx^B. \tag{6}$$

The lowercase Latin indices are lowered and raised using the metric g_{ab} and its inverse g^{ab}, respectively. Similarly, the capital Latin indices are raised and lowered using Ω_{AB} and Ω^{AB}. Since both g_{ab} and Ω_{AB} are symmetric matrices, we will not worry about the spaces in the index placement and write, for example, g^b_a instead of ${g_a}^b$ and Ω^B_A instead of ${\Omega_A}^B$.

2.2 Linear perturbations

Suppose now that the metric $g_{\mu\nu}$ is linearly perturbed by a small quantity $\gamma_{\mu\nu}$ (i.e. $|\gamma_{\mu\nu}| \ll 1$), such that the perturbed metric $g^{\mathrm{p}}_{\mu\nu}$ and its inverse take the form[2]

$$\begin{aligned} g^{\mathrm{p}}_{\mu\nu} &= g_{\mu\nu} + \gamma_{\mu\nu}, \qquad (8) \\ g^{\mathrm{p}\,\mu\nu} &= g^{\mu\nu} - \gamma^{\mu\nu}. \qquad (9) \end{aligned}$$

The perturbing metric $\gamma_{\mu\nu}$ and its inverse account for perturbations in both the (t, r)-components and the (θ, ϕ)-components. We make no assumptions regarding the nature of the perturbation. In particular, we allow for the perturbing metric to have cross terms:

$$\gamma_{\mu\nu} = \begin{bmatrix} \gamma_{ab} & \gamma_{aB} \\ \gamma_{Ab} & \gamma_{AB} \end{bmatrix}, \qquad \gamma^{\mu\nu} = \begin{bmatrix} \gamma^{ab} & \frac{1}{r^2}\gamma^{aB} \\ \frac{1}{r^2}\gamma^{Ab} & \frac{1}{r^4}\gamma^{AB} \end{bmatrix}. \tag{10}$$

[1] In Section 2.6 it will become clear that we are allowed to replace the mode variables by their gauge-invariant versions. The tilde " ˜ " is then dropped for notational brevity.

[2] There is a minus sign in the inverse perturbed metric since

$$g^{\mathrm{p}}_{\mu\nu} g^{\mathrm{p}\,\mu\lambda} := \delta^\lambda_\nu, \tag{7}$$

which holds to first order in $\gamma_{\mu\nu}$ only if we take the above definitions.

Just like the background metric $g_{\mu\nu}$, the perturbing metric is symmetric ($\gamma_{\mu\nu} = \gamma_{\nu\mu}$). The components of the inverse perturbing metric are obtained by raising the indices of $\gamma_{\mu\nu}$ with the background metric. For example:

$$g^{ab} g^{AB} \gamma_{bA} = g^{ab} \frac{1}{r^2} \Omega^{AB} \gamma_{bA} = \frac{1}{r^2} \gamma^{aB}. \tag{11}$$

Clearly, we have

$$\begin{aligned} {}^4g^{\mathrm{p}}_{ab} &= g_{ab} + \gamma_{ab}, \\ {}^4g^{\mathrm{p}}_{aB} &= \gamma_{aB}, \\ {}^4g^{\mathrm{p}}_{AB} &= r^2 \Omega_{AB} + \gamma_{AB}, \end{aligned} \tag{12}$$

$$\begin{aligned} {}^4g^{\mathrm{p}\,ab} &= g^{ab} - \gamma^{ab}, \\ {}^4g^{\mathrm{p}\,aB} &= -\frac{1}{r^2} \gamma^{aB}, \\ {}^4g^{\mathrm{p}\,AB} &= \frac{1}{r^2} \Omega^{AB} - \frac{1}{r^4} \gamma^{AB}. \end{aligned} \tag{13}$$

where the superscript "^{4}" indicates that the metric belongs to $\mathcal{M}^4$. This way we avoid confusion between the ab-component of $g_{\mu\nu}$ and g_{ab}, a component of the metric of $\mathcal{M}^2$. Since both submanifolds are two-dimensional, a superscript "^{2}" would be ineffective. We therefore write quantities defined with respect to g_{ab} and Ω_{AB} without this additional index.

2.3 Basic definitions

In this formalism, where linear perturbations are introduced and the metric is split, we can derive the key quantities needed for the study of metric perturbations. We begin by listing the covariant derivatives associated with each (sub-)metric, followed by the relevant curvature quantities (Christoffel symbols, Riemann tensor, Ricci tensor, and Ricci scalar). We discuss the proper way to commute covariant derivatives of the submanifolds and provide useful relations that help to simplify calculations in Schwarzschild coordinates.

2.3.1 Covariant derivatives

Covariant differentiation is defined independently on each manifold, requiring us to define separate covariant derivative for each metric. We denote these covariant derivatives as summarized in Table 1.

Manifold	$\mathcal{M}^4 = \mathcal{M}^2 \times \mathcal{S}^2$	$\mathcal{M}^2$	$\mathcal{S}^2$
Metric	$g_{\mu\nu}$	g_{ab}	Ω_{AB}
Covariant derivative	∇_μ	$\mathscr{D}_a$	D_A
Commutes with	-	D_A	$\mathscr{D}_a$

Table 1: A list of the symbols with which we indicate the manifolds and their respective metrics and covariant derivatives. By definition, the covariant derivatives are compatible with their respective metrics, meaning $\mathscr{D}_a g_{bc} = 0$ and $D_A \Omega_{BC} = 0$.

The covariant derivative $\mathscr{D}_a$ commutes with D_A because Ω_{AB} does not depend on t or r, and g_{ab} is independent of θ or ϕ. Furthermore, any quantity that depends only on x^a is covariantly constant with respect to the Christoffel symbols associated with $\mathscr{D}_a$, and quantities depending solely only on x^A are covariantly constant with respect to the Christoffel symbols of D_A. This means for example that

$$\begin{aligned} &D_A r := 0, \\ &D_A g_{ab} := 0, \\ &\mathscr{D}_a \Omega_{AB} := 0. \end{aligned} \tag{14}$$

2.3.2 Christoffel symbols

The Christoffel symbols belonging to ∇_μ are ${}^4\Gamma^\lambda_{\mu\nu}$. It is straightforward to show using the definition of the Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu} := \frac{1}{2} g^{\lambda\alpha} \left[\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right], \tag{15}$$

and the components of the full metric, Eq. (5), that the only non-zero components are

$$\begin{aligned} {}^4\Gamma^a_{bc} &= \Gamma^a_{bc}, \\ {}^4\Gamma^a_{BC} &= -r r^a \Omega_{BC}, \\ {}^4\Gamma^A_{Bc} &= \frac{r_c}{r}\delta^A_B, \\ {}^4\Gamma^A_{BC} &= \Gamma^A_{BC}. \end{aligned} \tag{16}$$

Note that Christoffel symbols with lowercase Latin indices *always* belong to $\mathscr{D}_a$, and ones with capital Latin indices belong to D_A. In Schwarzschild coordinates, the Christoffel symbols are explicitly

$$\begin{aligned} {}^4\Gamma^t_{rt} &= -{}^4\Gamma^r_{rr} = \frac{M}{r^2} f(r)^{-1}, \\ {}^4\Gamma^r_{tt} &= \frac{M}{r^2} f(r), \\ {}^4\Gamma^\phi_{\theta\phi} &= \cot\theta, \\ {}^4\Gamma^r_{\theta\theta} &= \frac{1}{\sin^2\theta} {}^4\Gamma^r_{\phi\phi} = -r f(r), \\ {}^4\Gamma^\theta_{\theta r} &= \frac{1}{\sin^2\theta} {}^4\Gamma^\theta_{\phi\phi} = {}^4\Gamma^\phi_{\phi r} = \frac{1}{r}. \end{aligned} \tag{17}$$

2.3.3 Riemann tensors, Ricci tensors and Ricci scalars

The non-zero components of the full Riemann tensor (${}^4R_{\mu\nu\rho\sigma}$), Ricci tensor (${}^4R_{\mu\nu}$) and the Ricci scalar (${}^4R$) are given in [36], but we will not make use of them in this thesis. We do however make use of the Riemann tensors of the submanifolds[3]:

$$\mathscr{R}_{abcd} = \frac{\mathscr{R}}{d(d-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{18}$$

$$R_{ABCD} = \Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}. \tag{19}$$

Here, $\mathscr{R}$ is the Ricci scalar of the submanifold $\mathcal{M}^d$. For the submanifold $\mathcal{M}^2$, the Ricci tensor is

$$\mathscr{R} = \frac{4M}{r^3}, \tag{20}$$

such that

$$\mathscr{R}_{abcd} = \frac{2M}{r^3}(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{21}$$

We should remark that (18) is only valid for spacetimes with constant curvature. This form of the Riemann tensor follows from the fact that a double contraction of its indices in a general dimension d should yield the Ricci scalar:

$$g^{ac}g^{bd}\mathscr{R}_{abcd} = \mathscr{R} g^{ac}g^{bd}(g_{ac}g_{bd} - g_{ad}g_{bc}) = \mathscr{R}(d^2 - d). \tag{22}$$

Clearly we have to divide by $d(d-1)$ to obtain the Ricci scalar.

[3]These Riemann tensors are *not the components* of the Riemann tensor belonging to $g_{\mu\nu}$, but belong to the submetrics g_{ab} and Ω_{AB}.

Note that in general for a two-dimensional manifold

$$\mathscr{R}_{ab} = \frac{\mathscr{R}}{2} g_{ab}, \tag{23}$$

based on symmetry arguments[4] (the Riemann tensor has only one independent component in two dimensions). In higher dimensions this relation does not hold, since the Riemann tensor will have more independent components.

2.3.4 Commuting covariant derivatives of the submanifolds

It is important to note that $\mathscr{D}_a$ and D_A do *not* commute with themselves. When we commute two covariant derivatives belonging to the same submanifold, a Riemann tensor term will appear (which is, indeed, a way to define curvature) [9]. This is readily seen by considering the action of a commutator of covariant derivatives of $\mathcal{M}^2$ on a tensor X (of arbitrary rank) in the absence of torsion [37]:

$$\begin{aligned} [\mathscr{D}_a, \mathscr{D}_b]\, X^{c_1\cdots c_k}{}_{d_1\cdots d_l} = &+ \mathscr{R}^{c_1}{}_{mab} X^{mc_2\cdots c_k}{}_{d_1\cdots d_l} + \mathscr{R}^{c_2}{}_{mab} X^{c_1 m\cdots c_k}{}_{d_1\cdots d_l} + \cdots \\ &- \mathscr{R}^{m}{}_{d_1ab} X^{c_1\cdots c_k}{}_{md_2\cdots d_l} - \mathscr{R}^{m}{}_{d_2ab} X^{c_1\cdots c_k}{}_{d_1 m\cdots d_l} - \cdots . \end{aligned} \tag{24}$$

An extra term appears for each additional index on X.

In this thesis, we will be concerned only with vectors and rank-2 tensors. When commuting two covariant derivatives acting on a co-vector v_a, (24) tells us that

$$\begin{aligned} \mathscr{D}_a\mathscr{D}_b v_c - \mathscr{D}_b\mathscr{D}_a v_c &= -\mathscr{R}^{d}{}_{cab} v_d \\ &= \mathscr{R}_{abc}{}^{d} v_d, \end{aligned} \tag{25}$$

while for a rank-2 tensor t_{ab}, the commutation gives

$$\begin{aligned} \mathscr{D}_a\mathscr{D}_b t_{cd} - \mathscr{D}_b\mathscr{D}_a t_{cd} &= -\mathscr{R}^{m}{}_{cab} t_{md} - \mathscr{R}^{m}{}_{dab} t_{cm} \\ &= \mathscr{R}_{abc}{}^{m} t_{md} + \mathscr{R}_{abd}{}^{m} t_{cm}. \end{aligned} \tag{26}$$

Here, we have utilized the symmetries of the Riemann tensor to arrange the indices conveniently. When commuting covariant derivatives belonging to $\mathcal{S}^2$, we obtain a similar expression:

$$D_A D_B v_C - D_B D_A v_C = R_{ABC}{}^{D} v_D, \tag{27}$$

$$D_A D_B t_{CD} - D_B D_A t_{CD} = R_{ABC}{}^{M} t_{MD} + R_{ABD}{}^{M} t_{CM}. \tag{28}$$

2.3.5 Useful relations

To maintain covariance in our calculations, we introduce the covector $r_a := \mathscr{D}_a r = \partial_a r = (0, 1)$, which enables us to express the function $f(r)$ in a covariant manner as

$$r^a r_a = g^{rr} = f(r). \tag{29}$$

On the other hand, when working in Schwarzschild coordinates we can make use of the fact that

$$\mathscr{D}_a r_b = \frac{M}{r^2} g_{ab}, \tag{30}$$

$$\Box r = \frac{2M}{r^2}. \tag{31}$$

The four-dimensional d'Alembertian operator is defined as $\Box := g^{ab}\mathscr{D}_a\mathscr{D}_b$. Note that relation (30) only holds in four dimensions.

[4] The same formula can also be obtained by contracting the first and third indices of the Riemann tensor, Eq. (18), and using $d = 2$. This however only holds if the spacetime has constant curvature. Equation (21) is *generally* true for a two-dimensional manifold. Coincidentally, the two are the same.

2.4 Linearized curvature quantities

In this section, we derive the perturbed curvature quantities. We begin by linearizing the Christoffel symbols, which allows us to compute the linearized Riemann tensor, Ricci tensor, Ricci scalar, and Einstein tensor. Subsequently, we evaluate the Ricci and Einstein tensors within the $\mathcal{M}^2 \times \mathcal{S}^2$-decomposition.

The four-dimensional vacuum Einstein equations[5] are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{32}$$

A contraction with $g^{\mu\nu}$ shows that $R = 0$ and hence

$$R_{\mu\nu} = 0. \tag{33}$$

This means that perturbations of the background spacetime are fully described by the components of the perturbed Ricci tensor,

$$\delta R_{\mu\nu} = 0. \tag{34}$$

As we will see in Section 2.8, it is more practical to work with the full Einstein tensors in the even-parity sector and use

$$\delta G_{\mu\nu} = 0. \tag{35}$$

We will first derive the covariant form of the perturbed Ricci tensor and Einstein tensor in terms of $\gamma_{\mu\nu}$. To this end, we first need the perturbed Christoffel symbols:

$${}^{\mathrm{p}}\Gamma^{\lambda}_{\mu\nu} = {}^{4}\Gamma^{\lambda}_{\mu\nu} + \delta\Gamma^{\lambda}_{\mu\nu}. \tag{36}$$

The linearized connection is [16]

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\nabla_\mu \gamma_{\alpha\nu} + \nabla_\nu \gamma_{\alpha\mu} - \nabla_\alpha \gamma_{\mu\nu} \right). \tag{37}$$

We omit the superscript “ ^{4} ” on linearized quantities like $\delta\Gamma^{\lambda}_{\mu\nu}$, as we understand that the perturbation is always carried out on the full background spacetime. The components of $\delta\Gamma^{\lambda}_{\mu\nu}$ are calculated using the first-order covariant derivatives of a rank-2 tensor in Eqs. (306) in Appendix C, and are given by

$$\begin{aligned}
\delta\Gamma^{a}_{bc} &= C^{a}_{bc}, \\
\delta\Gamma^{a}_{bC} &= \frac{1}{2} g^{ac} \left(\mathscr{D}_b \gamma_{cC} + D_C \gamma_{cb} - \mathscr{D}_c \gamma_{bC} \right) - \frac{1}{r} r_b \gamma^{a}_{C}, \\
\delta\Gamma^{a}_{BC} &= \frac{1}{2} (D_B \gamma^{a}_{C} + D_C \gamma^{a}_{B} - \mathscr{D}^a \gamma_{BC}) - r r_b \Omega_{BC} \gamma^{ab}, \\
\delta\Gamma^{A}_{bc} &= \frac{1}{2r^2} (\mathscr{D}_b \gamma^{A}_{c} + \mathscr{D}_c \gamma^{A}_{b} - D^A \gamma_{bc}), \\
\delta\Gamma^{A}_{bC} &= \frac{1}{2r^2} (\mathscr{D}_b \gamma^{A}_{C} + D_C \gamma^{A}_{b} - D^A \gamma_{bC}) - \frac{1}{r^3} r_b \gamma^{A}_{C}, \\
\delta\Gamma^{A}_{BC} &= \frac{1}{r^2} C^{A}_{BC} + \frac{1}{r} r_a \Omega_{BC} \gamma^{Aa},
\end{aligned} \tag{38}$$

where

$$C^{a}_{bc} := \frac{1}{2} \left(\mathscr{D}_c \gamma^{a}_{b} + \mathscr{D}_b \gamma^{a}_{c} - \mathscr{D}^a \gamma_{bc} \right), \tag{39}$$

$$C^{A}_{BC} := \frac{1}{2} \left(D_C \gamma^{A}_{B} + D_B \gamma^{A}_{C} - D^A \gamma_{BC} \right). \tag{40}$$

[5] In the calculations that will follow, we could include a source term in the form of a stress-energy tensor $T_{\mu\nu}$. This would complicate the calculations significantly. For a clear account of how to include such a source term in the calculations, see [21].

Similarly, the Ricci tensor of the perturbed spacetime is

$$ {}^{\mathrm{p}}R_{\mu\nu} = {}^{4}R_{\mu\nu} + \delta R_{\mu\nu}. \tag{41} $$

A general expression for the linearly perturbed Riemann tensor is[6] [38]

$$ \delta R_{\mu\nu\rho}{}^{\lambda} = -2\nabla_{[\mu}\left(\delta\Gamma^{\lambda}_{\nu]\rho}\right), \tag{44} $$

and the linearized Ricci tensor is simply its contraction of the upper and middle-lower indices,

$$ \delta R_{\mu\nu} := -2\nabla_{[\nu}\left(\delta\Gamma^{\lambda}_{\lambda]\mu}\right), \tag{45} $$

and the linearized Ricci scalar is

$$ \begin{aligned} \delta\left(g^{\mu\nu}R_{\mu\nu}\right) &= \left(\delta g^{\mu\nu}\right)R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} && (46)\\ &= g^{\mu\nu}\delta R_{\mu\nu} && (47) \end{aligned} $$

where we used that the Ricci tensor of the background vanishes in a vacuum spacetime. We can now express the perturbed curvature quantities in terms of the perturbing metric using Eq. (37):

$$ \begin{aligned} &\delta R_{\mu\nu\rho}{}^{\lambda} = -\nabla_{[\mu}\nabla_{\nu]}\gamma^{\lambda}_{\rho} - \nabla_{[\mu}\nabla_{|\rho|}\gamma^{\lambda}_{\nu]} + \nabla_{[\mu}\nabla^{\lambda}\gamma_{\nu]\rho}, && (48)\\ &\delta R_{\mu\nu} = -\frac{1}{2}{}^{4}\Box\gamma_{\mu\nu} - \frac{1}{2}\nabla_{\mu}\nabla_{\nu}\gamma^{\lambda}_{\lambda} + \nabla_{\lambda}\nabla_{(\mu}\gamma^{\lambda}_{\nu)}, && (49)\\ &\delta R = -{}^{4}\Box\gamma^{\mu}_{\mu} + \nabla_{\mu}\nabla_{\nu}\gamma^{\mu\nu}, && (50)\\ &\delta G_{\mu\nu} = -\frac{1}{2}{}^{4}\Box\gamma_{\mu\nu} + \nabla_{\lambda}\nabla_{(\mu}\gamma^{\lambda}_{\nu)} - \frac{1}{2}\nabla_{\nu}\nabla_{\mu}\gamma^{\lambda}_{\lambda} - \frac{1}{2}g_{\mu\nu}\left(\nabla^{\rho}\nabla^{\lambda}\gamma_{\lambda\rho} - \Box\gamma^{\lambda}_{\lambda}\right), && (51) \end{aligned} $$

where we denoted the antisymmetrization and symmetrization with square brackets and round brackets respectively, and vertical bars $|\cdot|$ indicate that we exclude the index $\cdot$ from the (anti-)symmetrization process. We identify the four-dimensional d'Alembertian operator as ${}^{4}\Box := \nabla^{\mu}\nabla_{\mu}$.

[6]The antisymmetrization of a rank-2 tensor is defined as

$$ T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}), \tag{42} $$

and the symmetrization as

$$ T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}). \tag{43} $$

By evaluating Eqs. (49) and (51) in the $\mathcal{M}^2 \times \mathcal{S}^2$-split, making use of the MATHEMATICA script 4D_PERT_COVARIANT.NB, we are able to reproduce the results in Appendix D of Martel and Poisson [21]. The components of the linearized Ricci tensor are found to be:

$$\begin{aligned}\delta R_{ab} = {} & \mathscr{D}_m C^m_{ab} + \frac{2}{r} r_m C^m_{ab} - \frac{1}{2}\mathscr{D}_a\mathscr{D}_b\gamma^m_m - \frac{1}{2r^2}D^M D_M\gamma_{ab} + \frac{1}{2r^2}D_M\left(\mathscr{D}_a\gamma^M_b + \mathscr{D}_b\gamma^M_a\right) \\ & - \frac{1}{2r^2}\mathscr{D}_a\mathscr{D}_b\gamma^M_M + \frac{1}{2r^3}\left(r_a\mathscr{D}_b\gamma^M_M + r_b\mathscr{D}_a\gamma^M_M\right) - \frac{1}{r^4}\left(r_a r_b - r\mathscr{D}_a r_b\right)\gamma^M_M, \end{aligned} \tag{52}$$

$$\begin{aligned}\delta R_{aB} = {} & \frac{1}{2}D_B\left(\mathscr{D}_m\gamma^m_a - \mathscr{D}_a\gamma^m_m + \frac{1}{r}r_a\gamma^m_m\right) - \frac{1}{2}\left(\Box\gamma_{aB} - \mathscr{D}_m\mathscr{D}_a\gamma^m_B\right) \\ & - \frac{1}{r}\left(r_a\mathscr{D}_m\gamma^m_B - r_m\mathscr{D}_a\gamma^m_B\right) - \frac{1}{r^2}\left(r_a r_m + r\mathscr{D}_a r_m\right)\gamma^m_B + \frac{1}{2r^2}D^M\left(D_B\gamma_{aM} - D_M\gamma_{aB}\right) \\ & - \frac{1}{2r^2}\mathscr{D}_a\left(D_M\gamma^M_B - D_B\gamma^M_M\right) - \frac{1}{r^3}r_a\left(D_M\gamma^M_B - D_B\gamma^M_M\right), \end{aligned} \tag{53}$$

$$\begin{aligned}\delta R_{AB} = {} & \Omega_{AB}\left[r r_a\mathscr{D}_b\left(\gamma^{ab} - \frac{1}{2}g^{ab}\gamma^m_m\right) + \left(r_a r_b + r\mathscr{D}_a\mathscr{D}_b r\right)\gamma^{ab}\right] - \frac{1}{2}D_A D_B\gamma^a_a \\ & + \frac{1}{2}\mathscr{D}_a\left(D_A\gamma^a_B + D_B\gamma^a_A\right) + \frac{1}{r}r_a\Omega_{AB}D_M\gamma^{aM} - \frac{1}{2}\Box\gamma_{AB} + \frac{1}{r^2}D_M C^M_{AB} \\ & - \frac{1}{2r^2}D_A D_B\gamma^M_M + \frac{1}{r}r^a\mathscr{D}_a\left(\gamma_{AB} - \frac{1}{2}\Omega_{AB}\gamma^M_M\right) - \frac{2}{r^2}r^a r_a\left(\gamma_{AB} - \frac{1}{2}\Omega_{AB}\gamma^M_M\right). \end{aligned} \tag{54}$$

The components of $\delta G_{\mu\nu}$ are:

$$\begin{aligned}\delta G_{ab} = {} & \frac{1}{2}g_{ab}\Box\gamma^c_c - \frac{1}{2}\mathscr{D}_b\mathscr{D}_a\gamma^c_c + \frac{2}{r}r^c\mathscr{D}_{(a}\gamma_{b)c} + \frac{1}{r}r^c\left(g_{ab}\mathscr{D}_c\gamma^d_d - \mathscr{D}_c\gamma_{ab}\right) + \mathscr{D}_c\mathscr{D}_{(a}\gamma^c_{b)} - \frac{1}{2}\Box\gamma_{ab} \\ & - g_{ab}\left(\frac{1}{r}(\mathscr{D}_d r_c)\gamma^{cd} + \frac{2}{r}r_c\mathscr{D}_d\gamma^{cd} + \frac{1}{r^2}r_c r_d\gamma^{cd} - \frac{1}{2}\mathscr{D}_d\mathscr{D}_c\gamma^{cd}\right) + \frac{1}{2r^2}D^A D_A\left(g_{ab}\gamma^c_c - \gamma_{ab}\right) \\ & + \frac{1}{r^3}\left(2r_{(a}D_A\gamma^A_{b)} - g_{ab}r^c D_A\gamma^A_c\right) + D_A\mathscr{D}_{(a}\gamma^A_{b)} - \frac{1}{r^2}g_{ab}D_A\mathscr{D}_c\gamma^{cA} \\ & + \frac{1}{2}g_{ab}\left(\Box\gamma^A_A - \frac{1}{r^4}D_B D_A\gamma^M_M + \frac{1}{r^2}D^B D_B\gamma^A_A\right) + \frac{r^2-1}{r^4}r_a r_b\gamma^A_A \\ & + \frac{1}{2r^3}g_{ab}\left(\Box r - \frac{1}{r}r^c r_c\right)\gamma^A_A + \frac{2r^2+1}{2r^3}g_{ab}r^c\mathscr{D}_c\gamma^A_A - \frac{1}{r}r_{(a}\mathscr{D}_{b)}\gamma^A_A - \frac{1}{2}\mathscr{D}_b\mathscr{D}_a\gamma^A_A, \end{aligned} \tag{55}$$

$$\begin{aligned}\delta G_{aB} = {} & \frac{1}{2r}r_a D_B\gamma^b_b - D_B\mathscr{D}_{[a}\gamma^b_{b]} + \frac{1-r^2}{r^2}r^b r_{(a}\gamma_{b)B} + \frac{r^2+1}{2r}r^b\mathscr{D}_a\gamma_{bB} - \frac{1}{r}(\mathscr{D}_b r_a)\gamma^b_B \\ & - \frac{1}{r}r_a\mathscr{D}_b\gamma^b_B + \frac{1}{2}\mathscr{D}_b\mathscr{D}_a\gamma^b_B + \frac{1-r}{2r}(\Box r)\gamma_{aB} - \frac{1}{2}\Box\gamma_{aB} + \frac{1}{2}D_A D_B\gamma^A_a \\ & - \frac{1}{2r^2}D^A D_A\gamma_{aB} + \frac{1-r^2}{r^3}r_a D_A\gamma^A_B + \mathscr{D}_a D_{[A}\gamma^A_{B]}, \end{aligned} \tag{56}$$

$$\begin{aligned}\delta G_{AB} = {} & r r^a\Omega_{AB}\left(\frac{1}{2}\mathscr{D}_a\gamma^b_b - \mathscr{D}_b\gamma^b_a\right) - \frac{1}{2}r^2\Omega_{AB}\mathscr{D}_b\mathscr{D}_a\gamma^{ab} + \frac{1}{2}\Omega_{AB}\left(r^2\Box\gamma^a_a + D^M D_M\gamma^a_a\right) \\ & - \frac{1}{2}D_B D_A\gamma^a_a + \mathscr{D}_a D_{(A}\gamma^a_{B)} - \Omega_{AB}D_M\mathscr{D}_b\gamma^{bM} + \frac{r^2-1}{r}r^a\left(D_{(A}\gamma_{B)a} + \Omega_{AB}D_M\gamma^M_a\right) \\ & - 2r^a r_a\gamma_{AB} - \frac{1}{2r^2}r^a r_a\Omega_{AB}\gamma^M_M + \frac{1-r^2}{r}(\Box r)\gamma_{AB} + \frac{1}{2r}(\Box r)\Omega_{AB}\gamma^M_M + \frac{1}{r}r^a\mathscr{D}_a\gamma_{AB} \\ & + \frac{r^2+1}{2r}\Omega_{AB}r^a\mathscr{D}_a\gamma^M_M - \frac{1}{2}\Box\gamma_{AB} + \frac{1}{2}r^2\Omega_{AB}\Box\gamma^M_M + D_M D_{(A}\gamma^M_{B)} - \frac{1}{2}D_B D_A\gamma^M_M \\ & - \frac{1}{2r^2}\left(D^M D_M\gamma_{AB} + \Omega_{AB}D_N D_M\gamma^{MN}\right) + \frac{1}{2}\Omega_{AB}D^M D_M\gamma^N_N. \end{aligned} \tag{57}$$

We can verify these equations by performing the evaluation manually, which is done by substituting Eqs. (38) into Eq. (45). As pointed out by Spiers et al. [39], in doing so we must proceed with caution, especially with terms that involve contractions. These terms should first be expanded into a $d+2$ form. Only after this expansion can we assign the free indices to either $\mathcal{M}^2$ or $\mathcal{S}^2$. For illustration, consider the following contraction:

$$g^{\mu\nu}\nabla_\mu\gamma_{\nu\rho} = g^{ab}\nabla_a\gamma_{b\rho} + \frac{1}{r^2}\Omega^{AB}\nabla_A\gamma_{B\rho}$$

Only now are we are allowed to choose $\rho = c$:

$$g^{\mu\nu}\nabla_\mu\gamma_{\nu c} = g^{ab}\mathscr{D}_a\gamma_{bc} + \frac{1}{r^2}\Omega^{AB}D_A\gamma_{Bc} + \frac{2}{r}r^a\gamma_{ac} - \frac{1}{r^3}r_c\gamma^A_A.$$

2.5 Decomposition into spherical harmonics

So far, we have not yet taken advantage of the spherical symmetry of the Schwarzschild metric. This symmetry naturally lends itself to use a special class of functions, spherical harmonics, which are defined on $\mathcal{S}^2$ and have well-known properties. Spherical harmonics come in three versions; scalar, vector and tensor harmonics [21]. This decomposition introduces two parities – even and odd – which describe the behaviour of the functions under a coordinate transformation on $\mathcal{S}^2$. A detailed discussion of the properties of these harmonics can be found in Appendix B. Decomposing the metric perturbations into spherical harmonics offers a key advantage: the spherical symmetry of the background ensures that modes with different parity do not mix, allowing the perturbation equations to be derived independently for each parity [9].

The perturbing metric components are decomposed as follows:

$$\begin{aligned}
\gamma_{ab} &= \sum_{\ell,m} f^{\ell m}_{ab}(t,r)Y^{\ell m}, \\
\gamma_{aA} &= \sum_{\ell,m} \left\{ j^{\ell m}_a(t,r)Y^{\ell m}_A + h^{\ell m}_a(t,r)X^{\ell m}_A \right\}, \\
\gamma_{AB} &= \sum_{\ell,m} \left\{ r^2K^{\ell m}(t,r)\Omega_{AB}Y^{\ell m} + r^2G^{\ell m}(t,r)Y^{\ell m}_{AB} + h^{\ell m}_2(t,r)X^{\ell m}_{AB} \right\},
\end{aligned} \tag{58}$$

where

$$f^{\ell m}_{ab} = \begin{pmatrix} f(r)H^{\ell m}_0(t,r) & H^{\ell m}_1(t,r) \\ H^{\ell m}_1(t,r) & \frac{1}{f(r)}H^{\ell m}_2(t,r) \end{pmatrix}, \tag{59}$$

$$j^{\ell m}_a = \begin{pmatrix} j^{\ell m}_0(t,r) \\ j^{\ell m}_1(t,r) \end{pmatrix}, \tag{60}$$

$$h^{\ell m}_a = \begin{pmatrix} h^{\ell m}_0(t,r) \\ h^{\ell m}_1(t,r) \end{pmatrix}. \tag{61}$$

The even-parity spherical harmonics are $Y^{\ell m}$, $Y^{\ell m}_A$, $\Omega_{AB}Y^{\ell m}$ and $Y^{\ell m}_{AB}$. The odd-parity spherical harmonics are $X^{\ell m}_A$ and $X^{\ell m}_{AB}$. As a result, there are seven even-parity modes ($H^{\ell m}_0$, $H^{\ell m}_1$, $H^{\ell m}_2$, $j^{\ell m}_0$, $j^{\ell m}_1$, $K^{\ell m}$ and $G^{\ell m}$) and three odd-parity modes ($h^{\ell m}_0$, $h^{\ell m}_1$ and $h^{\ell m}_2$), which are all functions of x^a, meaning they are defined only on $\mathcal{M}^2$ [40]. We restrict our work to $\ell \geq 2$, because $\ell = 0$ and $\ell = 1$ are non-radiating and require special treatment [9].

At this point, Eqs. (58) can be inserted into Eqs. (52)-(54) and (55)-(57). However, before proceeding with this substitution, we first take advantage of the gauge freedom inherent to our theory.

2.6 Gauge transformations

In the spherical harmonic decomposition of our four-dimensional spacetime, there are ten mode components: $h_0^{\ell m}$, $h_1^{\ell m}$, $h_2^{\ell m}$, $H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$, $j_0^{\ell m}$, $j_1^{\ell m}$, $K^{\ell m}$ and $G^{\ell m}$. By exploiting the gauge freedom of our theory, we can eliminate three even-parity components and one odd-parity component [9]. In this section, we demonstrate how this simplification is achieved by imposing the so-called *Regge-Wheeler (RW) gauge*.

In general, we can find the gauge degrees of freedom (DOF) by considering an infinitesimal coordinate transformation generated by a vector field Ξ^μ (the *gauge vector*):

$$x^\mu \to x^\mu - \Xi^\mu. \tag{62}$$

It is well known that under such coordinate transformation, the change of the perturbation tensor field is the Lie derivative of that tensor field with respect to Ξ^μ [41]:

$$\gamma_{\mu\nu} \longrightarrow \gamma'_{\mu\nu} := \gamma_{\mu\nu} + \mathcal{L}_\Xi g_{\mu\nu} = \gamma_{\mu\nu} + g_{\lambda\nu}\nabla_\mu \Xi^\lambda + g_{\mu\lambda}\nabla_\nu \Xi^\lambda = \gamma_{\mu\nu} + 2\nabla_{(\mu}\Xi_{\nu)}. \tag{63}$$

In the formalism of the metric split, the gauge transformations are generated by a dual vector field $\Xi_\mu = (\Xi_a, \Xi_A)$. This means that the components of the perturbation field transform as

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \gamma_{ab} - 2\mathscr{D}_{(a}\Xi_{b)}, \tag{64}$$

$$\gamma_{aB} \longrightarrow \gamma'_{aB} = \gamma_{aB} - \nabla_a \Xi_B - \nabla_B \Xi_a = \gamma_{aB} - \mathscr{D}_a \Xi_B - D_B \Xi_a + \frac{2}{r} r_a \Xi_B, \tag{65}$$

$$\gamma_{AB} \longrightarrow \gamma'_{AB} = \gamma_{AB} - 2D_{(A}\Xi_{B)} - 2r r^c \Omega_{AB}\Xi_c, \tag{66}$$

where we have made use of Eqs. (304) from Appendix C. It can be shown that Eqs. (52)-(57) are all invariant under these transformations when the background Ricci tensor ${}^4R_{\mu\nu}$ vanishes [21] (as they should given the Stewart-Walker lemma [40]).

The gauge vector Ξ_μ can also be divided into vectors with even and odd parity:

$$\begin{aligned} \Xi_a &= \sum_{\ell,m} \xi_a^{\ell m} Y^{\ell m}, \\ \Xi_A &= \sum_{\ell,m} \left\{ \xi_2^{\ell m} Y_A^{\ell m} + \xi_3^{\ell m} X_A^{\ell m} \right\}, \end{aligned} \tag{67}$$

such that the even-parity modes are $\xi_0^{\ell m}$, $\xi_1^{\ell m}$ and $\xi_2^{\ell m}$, and the odd-parity mode is $\xi_3^{\ell m}$. These are all functions of x^a. We will now examine how the perturbing metric transforms under a gauge transformation in both the even and odd parity sectors.

2.6.1 Odd-parity gauge transformations

By the division of Eq. (67), the odd-parity gauge transformations are generated by the gauge vector $\Xi_\mu^{(\mathrm{odd})} = (0, \Xi_A^{(\mathrm{odd})})$, with

$$\Xi_A^{(\mathrm{odd})} = \sum_{\ell,m} \xi_3^{\ell m} X_A^{\ell m}. \tag{68}$$

Since $\Xi_\mu^{(\mathrm{odd})}$ contains one arbitrary function ($\xi_3^{\ell m}$), it can be used to gauge fix one of the odd parity metric perturbations. By substituting Eq. (68) into Eqs. (65) and (66), we find that the odd-parity sector of the perturbing metric transforms as follows:

$$\begin{aligned} \gamma_{aB} &\longrightarrow \gamma'_{aB} = h_a^{\ell m} X_B^{\ell m} - \mathscr{D}_a \xi_3^{\ell m} X_B^{\ell m} + \frac{2}{r} r_A \xi_3^{\ell m} X_B^{\ell m} := h_a'^{\ell m} X_B^{\ell m}, \\ \gamma_{AB} &\longrightarrow \gamma'_{AB} = h_2^{\ell m} X_{AB}^{\ell m} - 2\xi_3^{\ell m} D_{(A} X_{B)}^{\ell m} = h_2^{\ell m} X_{AB}^{\ell m} - 2\xi_3^{\ell m} X_{AB}^{\ell m} := h_2'^{\ell m} X_{AB}^{\ell m}. \end{aligned} \tag{69}$$

This means that

$$h_a^{\ell m} \longrightarrow h_a^{\prime \ell m} = h_a^{\ell m} - \mathscr{D}_a \xi_3^{\ell m} + \frac{2}{r} r_a \xi_3^{\ell m}, \tag{70}$$

$$h_2^{\ell m} \longrightarrow h_2^{\prime \ell m} = h_2^{\ell m} - 2\xi_3^{\ell m}. \tag{71}$$

Gauge-invariant quantities are derived by taking a linear combination of these transformation equations, ensuring that the terms involving $\xi_3^{\ell m}$ on the right-hand-side are eliminated. We denote these gauge-invariant quantities with a tilde " ˜ ":

$$\tilde{h}_a^{\ell m} = h_a^{\ell m} - \frac{1}{2}\mathscr{D}_a h_2^{\ell m} + \frac{1}{r} r_a h_2^{\ell m}. \tag{72}$$

Transforming these two quantities according to (70) indeed shows that they are unchanged.

Eq. (71) demonstrates that it is always possible to select a gauge where $h_2^{\ell m} = 0$ by choosing $\xi_3^{\ell m} = \frac{1}{2} h_2^{\ell m}$. This particular choice is referred to as the RW gauge. Setting $h_2^{\ell m} = 0$ in Eq. (72) leads to

$$\tilde{h}_0^{\ell m} = h_0^{\ell m}, \tag{73}$$

$$\tilde{h}_1^{\ell m} = h_1^{\ell m}, \tag{74}$$

indicating that, within the RW gauge, the modes $h_0^{\ell m}$ and $h_1^{\ell m}$ are equal to their gauge-invariant counterparts. Consequently, we can replace $h_0^{\ell m}$ and $h_1^{\ell m}$ by their gauge-invariant versions to recover gauge invariance of the final results[7] [9].

2.6.2 Even-parity gauge transformations

The even-parity gauge transformations are generated by the gauge vector $\Xi_\mu^{(\text{even})} = (\Xi_a^{(\text{even})}, \Xi_A^{(\text{even})})$, with

$$\begin{aligned} \Xi_a^{(\text{even})} &= \sum_{\ell,m} \xi_a^{\ell m} Y^{\ell m}, \\ \Xi_A^{(\text{even})} &= \sum_{\ell,m} \xi_2^{\ell m} Y_A^{\ell m}. \end{aligned} \tag{75}$$

This means that the even-parity gauge vector contains three arbitrary functions ($\xi_0^{\ell m}$, $\xi_1^{\ell m}$ and $\xi_2^{\ell m}$) that can be used to fix three components of the metric perturbations. Similarly as in the odd-parity case, we substitute Eqs. (75) into Eqs. (64), (65) and (66) to see that the even-parity modes transform as

$$f_{ab}^{\ell m} \longrightarrow f_{ab}^{\prime \ell m} = f_{ab}^{\ell m} - 2\mathscr{D}_{(a}\xi_{b)}^{\ell m}, \tag{76}$$

$$j_a^{\ell m} \longrightarrow j_a^{\prime \ell m} = j_a^{\ell m} - \xi_a^{\ell m} - \mathscr{D}_a \xi_2^{\ell m} + \frac{2}{r} r_a \xi_2^{\ell m}, \tag{77}$$

$$K^{\ell m} \longrightarrow K^{\prime \ell m} = K^{\ell m} + \frac{\ell(\ell+1)}{r^2}\xi_2^{\ell m} - \frac{2}{r} r^a \xi_a^{\ell m}, \tag{78}$$

$$G^{\ell m} \longrightarrow G^{\prime \ell m} = G^{\ell m} - \frac{2}{r^2}\xi_2^{\ell m}. \tag{79}$$

The gauge-invariant quantities are

$$\tilde{f}_{ab}^{\ell m} = f_{ab}^{\ell m} - \mathscr{D}_{(a}\left(j_{b)}^{\ell m} - \frac{r^2}{2}\mathscr{D}_{b)} G^{\ell m}\right), \tag{80}$$

$$\tilde{K}^{\ell m} = K^{\ell m} + \frac{\ell(\ell+1)}{2} G^{\ell m} - \frac{2}{r} r^a \left(j_a^{\ell m} - \frac{r^2}{2}\mathscr{D}_a G^{\ell m}\right). \tag{81}$$

[7] This is of course only possible if the final results themselves are gauge invariant. In a vacuum background spacetime, the linearized vacuum Einstein equations are gauge invariant by the Stewart-Walker lemma, we are allowed to replace h_0 and h_1 by $\tilde{h}_0$ and $\tilde{h}_1$ and call the final result gauge invariant [40].

By choosing $\xi_2^{\ell m} = \frac{r^2}{2} G^{\ell m}$ and $\xi_a^{\ell m} = j_a^{\ell m} - \frac{r^2}{2} \mathscr{D}_a G^{\ell m}$, we can set $j_a^{\ell m} = 0$ and $G^{\ell m} = 0$. This corresponds to the RW gauge in the even-parity sector. From Eqs. (80) and (81), it then follows that in the RW gauge,

$$\tilde{f}_{ab}^{\ell m} = f_{ab}^{\ell m}, \tag{82}$$

$$\tilde{K}^{\ell m} = K^{\ell m}. \tag{83}$$

With this knowledge, we are ready to compute the RW and Zerilli equations. From this point onward, we will omit the summation over ℓ and m on all the relevant quantities mentioned above for notational convenience (it will remain implicit). Additionally, we will drop the overhead tilde on the mode variables, since we can always substitute their gauge-invariant versions to recover gauge invariance of the final results.

2.7 Regge-Wheeler equation

The odd-parity sector of Eq. (58) in the RW gauge is

$$\begin{aligned} \gamma_{ab}^{(\text{odd})} &= 0, \\ \gamma_{aA}^{(\text{odd})} &= h_a(t,r) X_A, \\ \gamma_{AB}^{(\text{odd})} &= 0. \end{aligned} \tag{84}$$

To derive the RW equation, we first need to express Eqs. (52)-(54) in terms of the odd-parity harmonics from (84). A detailed calculation in terms of the dimension d is given in Appendix D. We now focus on the results for $d = 2$ and simplify the equations as much as possible. To this extent, we make use of the fact that

$$-\frac{1}{r} h_b \mathscr{D}_a \mathscr{D}^b r = -\frac{1}{r}\frac{M}{r^2} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = -\frac{M}{r^3} h_a \tag{85}$$

to simplify $\delta R_{aB}^{(\text{odd})}$ and obtain

$$\begin{aligned} \delta R_{ab}^{(\text{odd})} &= 0, \\ \delta R_{aB}^{(\text{odd})} &= 0 = \left[-\Box h_a + \mathscr{D}_a \mathscr{D}^b h_b + \frac{2}{r}\left(r^b \mathscr{D}_a h_b - r_a \mathscr{D}^b h_b \right) - \frac{2}{r^2} r_a r^b h_b + \frac{\ell(\ell+1)}{r^2} h_a \right] X_B, \\ \delta R_{AB}^{(\text{odd})} &= 0 = [\mathscr{D}^a h_a] X_{AB}. \end{aligned} \tag{86}$$

We have simply equated each component to zero to arrive at the vacuum Einstein equations, resulting in two coupled equations in the variables h_a. There are in principle two ways to decouple them:

1. we could either evaluate both equations in a specific coordinate system and solve them as a system of coupled PDEs (the brute force way),
2. or we can introduce two functions that decouple the system naturally in its covariant form (the clever way).

For the sake of completeness, we will work out both approaches, starting with method (1). The advantage of this approach is that the algebraic manipulations required are conceptually simple. However, as the equations tend to be lengthy, the calculations can be cumbersome, especially in the even-parity case where there are more equations and variables. Therefore, we will also explore method (2). The advantage of this approach is that it significantly reduces the length of the expressions we have to deal with. Covariant expressions also provide a clearer view of how the equations will behave when we extend them to higher dimensions. We will see later in this thesis however that also the covariant method is ponderous in the even-parity sector.

2.7.1 Decoupling in coordinates

We begin developing method (1) by evaluating Eqs. (86) in Schwarzschild coordinates. This evaluation is carried out using the MATHEMATICA script 4D_PERT_COORDINATES.NB, which produces the following results:

$$\begin{aligned}
\delta R^{(\text{odd})}_{tA} &= 0 = \frac{1}{2}\left[f(r)\left(-\partial_r^2 h_0 + \left[\partial_r + \frac{2}{r}\right]\partial_t h_1\right) + \left(\frac{\ell(\ell+1)}{r^2} - \frac{4M}{r^3}\right)h_0\right]X_A,\\
\delta R^{(\text{odd})}_{rA} &= 0 = \frac{1}{2}\left[-\frac{1}{f(r)}\left[\partial_r - \frac{2}{r}\right]\partial_t h_0 + \frac{(\ell-1)(\ell+2)}{r^2}h_1 + \frac{1}{f(r)}\partial_t^2 h_1\right]X_A,\\
\delta R^{(\text{odd})}_{AB} &= 0 = \left[-\frac{1}{f(r)}\partial_t h_0 + \partial_r\left[f(r)h_1\right]\right]X_{AB}.
\end{aligned} \tag{87}$$

It can be proven that the first equation is a consequence of the other two, and therefore does not provide any new information [40]. System (87) can be rewritten into a single equation for h_1. To demonstrate this, we first decouple the angular part from the equations, which is in this case trivial[8]. We can then rewrite the third equation as

$$\partial_t h_0 = f(r)\partial_r\left[f(r)h_1\right], \tag{88}$$

and substitute it in the second equation, which gives

$$-\frac{1}{f(r)}\partial_t^2 h_1 + \frac{1}{f(r)}\left(\partial_r - \frac{2}{r}\right)\left(f(r)\partial_r\left[f(r)h_1\right]\right) - \frac{(\ell-1)(\ell+2)}{r^2}h_1 = 0. \tag{89}$$

This result can be written more compactly by introducing a clever choice of function; the *Regge-Wheeler function*

$$\Psi_{\text{RW}} := \frac{f(r)}{r}h_1. \tag{90}$$

Eq. (89) can then be rewritten into the *Regge-Wheeler equation*, a wave equation with an associated potential:

$$\left(\Box - V_{\text{RW}}\right)\Psi_{\text{RW}} = 0, \tag{91}$$

$$V_{\text{RW}} := \frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}. \tag{92}$$

The d'Alembertian operator on $\mathcal{M}^2$ in Schwarzschild coordinates is given by

$$\begin{aligned}
\Box\Psi_{\text{RW}} :&= \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,g^{ab}\partial_b\Psi_{\text{RW}}\right)\\
&= \partial_r(g^{rr}\partial_r\Psi_{\text{RW}}) + \partial_t(g^{tt}\partial_t\Psi_{\text{RW}})\\
&= \left(-\frac{1}{f(r)}\partial_t^2 + f(r)\partial_r^2 + \frac{2M}{r^2}\partial_r\right)\Psi_{\text{RW}},
\end{aligned} \tag{93}$$

where $g := \det g_{ab} = -1$ is the determinant of g_{ab}.

The RW equation can be expressed as a one-dimensional Schrödinger-like equation. This is achieved by transforming from the radial coordinate r to the tortoise coordinate r^*, assuming an exponential time dependence. Specifically, we define the transformation as

$$\frac{dr}{dr^*} := f(r), \tag{94}$$

[8] The angular part has no dependency on the other coordinates (and vice versa), so it can be set to zero independently, making this part trivially satisfied. If a source term were present on the right-hand side, removing the spherical harmonic component of the field equations would be less trivial. In that case, the angular part is separated by integrating over the two-sphere and using orthonormality relations obeyed by the spherical harmonics, as discussed in [9].

such that

$$\partial^2_{r^*} = \frac{\partial}{\partial r^*}\left(\frac{dr}{dr^*}\frac{\partial}{\partial r}\right) = \frac{dr}{dr^*}\frac{\partial}{\partial r}\left(f(r)\frac{\partial}{\partial r}\right) = f(r)^2\partial_r^2 + f(r)[\partial_r f(r)]\partial_r. \tag{95}$$

Assuming the RW function has a time dependence of the form $e^{-i\omega t}$, we can write

$$\Psi_{\rm RW}(t,r) = \Psi_{\rm RW}(r)e^{-i\omega t}, \tag{96}$$

such that (91) can be written as

$$\frac{d^2\Psi_{\rm RW}}{dr^{*2}} + \left[\omega^2 - V_{\rm rw}(r^*)\right]\Psi_{\rm RW} = 0. \tag{97}$$

Note that the associated potential differs from (92) by a factor of $f(r)$:

$$V_{\rm rw}(r) := f(r)V_{\rm RW}(r). \tag{98}$$

This concludes the decoupling of the odd-parity equations in coordinates. The RW function in the form of Eq. (97) is needed to prove isospectrality in Section 2.9.1.

2.7.2 Decoupling covariantly

The system can be decoupled covariantly by introducing the covariant form of the RW function. This approach ensures that the equations remain in a form that is manifestly covariant, allowing for a more general analysis without needing to rely on specific choice of coordinates. The covariant RW function is[9]

$$\Psi_{\rm RW} := \frac{1}{r}r^a h_a. \tag{99}$$

Contracting the second equation in (86) with $r^{-1}r^a$, using Eq. (29) and using $\mathscr{D}^a h_a = 0$ (which follows from the third equation in (86)), we obtain

$$0 = \frac{1}{r}r^a\left(-\Box h_a + \frac{2}{r}r^b\mathscr{D}_a h_b\right) - \frac{1}{r^3}\left[2f(r) - \ell(\ell+1)\right]r^a h_a. \tag{100}$$

Our goal is to express this equation in terms of the covariant RW function. For the first term this can be done by bringing $r^{-1}r^a$ over the d'Alembertian operator towards h_a:

$$\begin{aligned}
\frac{1}{r}r^a\Box h_a &= g^{bc}\mathscr{D}_c\left(\frac{1}{r}r^a\mathscr{D}_b h_a\right) - g^{bc}(\mathscr{D}_b h_a)\mathscr{D}_c\left(\frac{1}{r}r^a\right)\\
&= g^{bc}\mathscr{D}_c\mathscr{D}_b\left(\frac{1}{r}r^a h_a\right) + \frac{2}{r^2}r^a r^b\mathscr{D}_b h_a - \frac{2}{r}(\mathscr{D}^b r^a)\mathscr{D}_b h_a - \frac{2}{r^3}r^a r^b r_b h_a\\
&\quad + \frac{2}{r^2}r_b(\mathscr{D}^b r^a)h_a + \frac{1}{r^2}r^a(\Box r)h_a - \frac{1}{r}(\Box r^a)h_a\\
&= \Box\left(\frac{1}{r}r^a h_a\right) - \frac{2}{r}(\mathscr{D}^b r^a)\mathscr{D}_b h_a - \frac{1}{r^3}r^a r^b r_b h_a + \frac{1}{r^2}r_b(\mathscr{D}^b r^a)h_a\\
&\quad + \frac{1}{r^2}r^a(\Box r)h_a - \frac{1}{r}(\Box r^a)h_a + \frac{2}{r}r^b\mathscr{D}_b\left(\frac{1}{r}r^a h_a\right)\\
&= \Box\Psi_{\rm RW} + \frac{2}{r}r^b\mathscr{D}_b\Psi_{\rm RW} + \frac{4M}{r^3}\Psi_{\rm RW}.
\end{aligned} \tag{101}$$

In the second step we worked out all derivatives, in the third step we used that

$$\frac{2}{r^2}r^a r^b\mathscr{D}_b h_a - \frac{2}{r^3}r^a r^b r_b h_a + \frac{2}{r^2}r_b(\mathscr{D}^b r^a)h_a = \frac{2}{r}r^b\mathscr{D}_b\left(\frac{1}{r}r^a h_a\right), \tag{102}$$

[9] Some authors (like Regge and Wheeler [16] themselves) define this function with a negative sign, which is purely conventional. We choose to adopt the definition without a minus sign as in Martel and Poisson [21].

and in the last step we inserted Eq. (99) and used Eq. (30) to show that

$$-\frac{2}{r}(\mathscr{D}^b r^a)\mathscr{D}_b h_a = -\frac{2M}{r^3}\underbrace{\mathscr{D}^a h_a}_{=0}. \tag{103}$$

We rewrite the second term in (100) as

$$\begin{aligned}\frac{2}{r^2}r^a r^b \mathscr{D}_a h_b &= \frac{2}{r}\left(r^a\mathscr{D}_a\left(\frac{1}{r}r^b h_b\right) - \frac{1}{r}r^a(\mathscr{D}_a r^b)h_b + \frac{1}{r^2}r^a r^b r_a h_b\right)\\ &= \frac{2}{r}\left(r^a\mathscr{D}_a\Psi_{\rm RW} - \frac{M}{r^3}r^a h_a + \frac{f(r)}{r^2}r^a h_a\right)\\ &= \frac{2}{r}r^a\mathscr{D}_a\Psi_{\rm RW} + \frac{2}{r^2}\left(1-\frac{3M}{r}\right)\Psi_{\rm RW},\end{aligned} \tag{104}$$

where we made use of relations (29) and (30) to go from the first to the second line. Substituting results (101) and (104) into Eq. (100) gives us exactly Eq. (91) with the correct potential.

This concludes the discussion of odd-parity perturbations of the Schwarzschild spacetime. It is evident that both methods require roughly the same amount of effort to derive the RW equation. In the next section, we will apply both approaches to the even-parity sector, aiming to obtain the Zerilli equation. This process will be considerably more challenging due to the additional variables and perturbation equations involved.

2.8 Zerilli equation

The approach to finding the Zerilli equation slightly differs from the one we adopted in the odd-parity sector. In Section 2.8.1, where we decouple the even-parity perturbation equations in coordinates, we make use of the components of $\delta R_{\mu\nu}$ in the vacuum Einstein equations, similarly to the odd-parity case. However, for covariant decoupling, which we explore in Section 2.8.2, we work with the components of $\delta G_{\mu\nu}$. The latter allows us to closely follow the calculations in Martel's Section 2.5.1 [9] and facilitate direct comparison with his results.

To derive the Zerilli equation, we first express Eqs. (52)-(54) in terms of the even-parity harmonics from (58) in the RW gauge, that is, in terms of

$$\begin{aligned}\gamma_{ab}^{\rm (even)} &= f_{ab}(t,r)Y,\\ \gamma_{aA}^{\rm (even)} &= 0,\\ \gamma_{AB}^{\rm (even)} &= r^2K(t,r)\Omega_{AB}Y.\end{aligned} \tag{105}$$

An explicit calculation is relegated to Appendix D. We find that the resulting vacuum Einstein equations are

$$\begin{aligned}
\delta R_{ab}^{(\text{even})} = 0 = &\left[\frac{1}{2}\mathscr{D}_b\mathscr{D}_m f_a^m + \frac{4M}{r^3}\left(f_{ba} - g_{ba}f_m^m\right) + \frac{1}{2}\mathscr{D}_a\mathscr{D}_m f_b^m - \frac{1}{2}\Box f_{ab}\right.\\
&+ \frac{1}{r}r_m\left(\mathscr{D}_b f_a^m + \mathscr{D}_a f_b^m - \mathscr{D}^m f_{ab}\right) - \frac{1}{2}\mathscr{D}_a\mathscr{D}_b f_m^m + \frac{\ell(\ell+1)}{2r^2}f_{ab}\\
&\left.- \frac{1}{r}r_b\mathscr{D}_a K - \frac{1}{r}r_a\mathscr{D}_b K - \mathscr{D}_a\mathscr{D}_b K\right]Y,\\
\delta R_{aB}^{(\text{even})} = 0 = &\frac{1}{2}\left[\mathscr{D}_b f_a^b - \mathscr{D}_a f_b^b + \frac{1}{r}r_a f_b^b - \mathscr{D}_a K\right]Y_B,\\
\delta R_{AB}^{(\text{even})} = 0 = &\left[r r_a\mathscr{D}_b f^{ab} - \frac{1}{2}r r^b\mathscr{D}_b f_m^m + r_a r_b f^{ab} + r\mathscr{D}_a r_b f^{ab} - \frac{1}{2}\Box(r^2K)\right.\\
&\left.+ \frac{1}{2}\ell(\ell+1)K + \frac{1}{4}\ell(\ell+1)f_a^a\right]\Omega_{AB}Y - \frac{1}{2}f_a^a Y_{AB}.
\end{aligned} \tag{106}$$

From the third equation, we can see that the terms multiplying $\Omega_{AB}Y$ and Y_{AB} have to be zero individually ($\Omega_{AB}Y$ and Y_{AB} are by definition orthogonal). This means that from the latter we obtain a condition on the trace of f_{ab}:

$$0 = f_a^a = \text{Tr}(f_{ab}) = -H_0 + H_2 \longrightarrow H_0 = H_2. \tag{107}$$

This allows us to eliminate f_a^a when working covariantly, or to remove either H_0 or H_2 when working in coordinates.

2.8.1 Decoupling in coordinates

In this section, we decouple the even-parity perturbation equations according to the method described by Zerilli[10] [43]. Similarly to the odd-parity sector, we start by evaluating the non-zero components of Eqs. (106) in Schwarzschild coordinates using the script 4D_PERT_COORDINATES.NB. Extracting the spherical harmonic functions and imposing the trace condition, we obtain a system of coupled PDEs:

$$\begin{aligned}
0 = &-\frac{1}{f(r)}\partial_t^2 K + \frac{M}{r^2}\partial_r K - \frac{1}{2f(r)}\partial_t^2 H_0 - \frac{f(r)}{2}\partial_r^2 H_0 - \frac{1}{r}\partial_r H_0 + \frac{\ell(\ell+1)}{2r^2}H_0\\
&+ \frac{2r-3M}{r^2 f(r)}\partial_t H_1 + \partial_t\partial_r H_1,\\
0 = &-\frac{r-3M}{r^2 f(r)}\partial_t K - \partial_t\partial_r K + \frac{1}{r}\partial_t H_0 + \frac{\ell(\ell+1)}{2r^2}H_1,\\
0 = &-f(r)\partial_r^2 K + \frac{1}{2f(r)}\partial_t^2 H_0 + \frac{f(r)}{2}\partial_r^2 H_0 + \frac{1}{r}\partial_r H_0 + \frac{\ell(\ell+1)}{2r^2}H_0 - \frac{M}{r^2 f(r)}\partial_t H_1\\
&- \partial_t\partial_r H_1 - \frac{2r-3M}{r^2}\partial_r K,\\
0 = &\,\partial_t K + \partial_t H_0 - f(r)\partial_r H_1 - \frac{2M}{r^2}H_1,\\
0 = &\,\partial_r K - \partial_r H_0 - \frac{2M}{r^2 f(r)}H_0 + \frac{1}{f(r)}\partial_t H_1,\\
0 = &\,\frac{1}{f(r)}\partial_t^2 K - f(r)\partial_r^2 K + \frac{6M-4r}{r^2}\partial_r K + \frac{(\ell+2)(\ell-1)}{r^2}K + \frac{2}{r}f(r)\partial_r H_0\\
&+ \frac{2}{r^2}H_0 - \frac{2}{r}\partial_t H_1.
\end{aligned} \tag{108}$$

[10] Special thanks to my fellow master student Tom van der Steen who pointed out how to derive the algebraic identity and how to decouple the system from Eqs. (123) onwards [42].

The first three equations follow from evaluating $\delta R_{tt}^{(\text{even})} = 0$, $\delta R_{tr}^{(\text{even})} = 0$ and $\delta R_{rr}^{(\text{even})} = 0$ respectively, the fourth and fifth equations follows from $\delta R_{tB}^{(\text{even})} = 0$ and $\delta R_{rB}^{(\text{even})} = 0$, and the last equation follows from the first part of $\delta R_{AB}^{(\text{even})} = 0$.

The odd-parity sector in coordinates consists of three coupled equations, given by Eqs. (87), while the even-parity sector consists of seven (Eqs. (108) and the trace condition). This means there are ten coupled perturbation equations in total, but only six gauge invariant variables. The total system is however not overdetermined; the Bianchi identities provide four additional constraints (three even-parity and one odd-parity) on the perturbation equations, reducing the number of independent equations to six. The Bianchi identities will not be utilized in this thesis. For reference, they are provided in [9].

Eqs. (108) reveals that the even-parity sector comprises three first-order PDEs and three second-order PDEs in three unknowns. As noted by Regge and Wheeler [16], the first-order equations alone do not provide sufficient information to solve the system. The non-trivial information contained in the second-order equations can be encapsulated in an "algebraic relation"[11], which is a third-order equation involving only time derivatives. With this algebraic relation, any of the second-order equations can be derived from the first-order ones. The algebraic relation plays a crucial role in reformulating the system into the Zerilli equation. Following the method introduced by Zerilli [43], the objective is to eliminate all r-derivatives from the second-order equations using the first-order ones, such that we obtain a single PDE that involves only time derivatives.

The first step in obtaining the algebraic relation is to observe the similarity between the first and third equations of (108); adding them gives

$$-\frac{1}{f(r)}\partial_t^2 K - f(r)\partial_r^2 K - \frac{2}{r}f(r)\partial_r K + \frac{\ell(\ell+1)}{r^2}H_0 + \frac{2}{r}\partial_t H_1 = 0. \tag{109}$$

Subtracting from this the last equation of (108) eliminates the term $-\partial_r^2 K$:

$$\begin{aligned} -\frac{2}{f(r)}\partial_t^2 K + \frac{2(r-M)}{r^2}\partial_r K - \frac{(\ell+2)(\ell-1)}{r^2}K \\ -\frac{2}{r}f(r)\partial_r H_0 + \frac{(\ell+2)(\ell-1)}{r^2}H_0 + \frac{4}{r}\partial_t H_1 = 0. \end{aligned} \tag{110}$$

We take a time derivative of this equation to obtain

$$\begin{aligned} -\frac{2}{f(r)}\partial_t^3 K + \frac{2(r-M)}{r^2}\partial_t\partial_r K - \frac{(\ell+2)(\ell-1)}{r^2}\partial_t K \\ -\frac{2}{r}f(r)\partial_t\partial_r H_0 + \frac{(\ell+2)(\ell-1)}{r^2}\partial_t H_0 + \frac{4}{r}\partial_t^2 H_1 = 0. \end{aligned} \tag{111}$$

Now note that the second equation in (108) can be written as

$$\partial_t\partial_r K = -\frac{r-3M}{r^2 f(r)}\partial_t K + \frac{1}{r}\partial_t H_0 + \frac{\ell(\ell+1)}{2r^2}H_1, \tag{112}$$

which we substitute in Eq. (111), yielding

$$\begin{aligned} -\frac{2}{f(r)}\partial_t^3 K - \frac{2(r-M)(r-3M)}{r^4 f(r)}\partial_t K + \frac{2(r-M)}{r^3}\partial_t H_0 + \frac{(\ell+2)(\ell-1)}{r^2}\partial_t H_0 \\ -\frac{(\ell+2)(\ell-1)}{r^2}\partial_t K - \frac{2}{r}f(r)\partial_t\partial_r H_0 + \frac{4}{r}\partial_t^2 H_1 + \frac{r-M}{r^2}\frac{\ell(\ell+1)}{r^2}H_1 = 0. \end{aligned} \tag{113}$$

[11] Regge and Wheeler worked in the Fourier domain, assuming a time dependence of the form $e^{-i\omega t}$ for the variables H_0, H_1 and K which effectively replaces the time derivatives with prefactors of $-i\omega t$. This gives the resulting equations an algebraic appearance.

Taking a time derivative of the fourth equation in (108) gives

$$\partial_t\partial_r K - \partial_t\partial_r H_0 - \frac{2M}{r^2 f(r)}\partial_t H_0 + \frac{1}{f(r)}\partial_t^2 H_1 = 0. \tag{114}$$

Substituting Eq. (112) into Eq. (114) gives

$$\partial_t\partial_r H_0 = -\frac{r-3M}{r^2 f(r)}\partial_t K + \frac{r-4M}{r^2 f(r)}\partial_t H_0 + \frac{1}{f(r)}\partial_t^2 H_1 + \frac{\ell(\ell+1)}{2r^2}H_1. \tag{115}$$

Finally, inserting Eq. (115) into Eq. (113) yields the desired algebraic relation:

$$\begin{aligned}\frac{2}{f(r)}\partial_t^3 K + \left[\frac{(\ell+2)(\ell-1)}{r^2} + \frac{2M(r-3M)}{r^4 f(r)}\right]\partial_t K \\ - \left[\frac{(\ell+2)(\ell-1)}{r^2} + \frac{6M}{r^3}\right]\partial_t H_0 + \frac{2}{r}\partial_t^2 H_1 - \frac{\ell(\ell+1)}{r^4}MH_1 = 0.\end{aligned} \tag{116}$$

We now have a set of four equations – comprising the first-order equations from (108) and the algebraic identity (116) – that contain all the information of the system. For clarity, we reiterate them:

$$\begin{aligned}
0 &= -\frac{r-3M}{r^2 f(r)}\partial_t K - \partial_t\partial_r K + \frac{1}{r}\partial_t H_0 + \frac{\ell(\ell+1)}{2r^2}H_1,\\
0 &= \partial_t K + \partial_t H_0 - f(r)\partial_r H_1 - \frac{2M}{r^2}H_1,\\
0 &= \partial_r K - \partial_r H_0 - \frac{2M}{r^2}\frac{1}{f(r)}H_0 + \frac{1}{f(r)}\partial_t H_1,\\
0 &= \frac{2}{f(r)}\partial_t^3 K + \left[\frac{(\ell+2)(\ell-1)}{r^2} + \frac{2M(r-3M)}{r^4 f(r)}\right]\partial_t K\\
&\quad - \left[\frac{(\ell+2)(\ell-1)}{r^2} + \frac{6M}{r^3}\right]\partial_t H_0 + \frac{2}{r}\partial_t^2 H_1 - \frac{\ell(\ell+1)}{r^4}MH_1.
\end{aligned} \tag{117}$$

It can be shown that third equation in (117) is consistent with the others, so that all information is stored in the first, second and fourth equations[12] [18]. It remains to show that Eqs. (117) can be decoupled and written as the Zerilli equation.

We solve the algebraic identity for $\partial_t H_0$ and substitute it into the first and second equations of (117). This results in a system of two coupled equations:

$$\begin{aligned}
\partial_t\partial_r K &= \alpha_0(r)\partial_t^3 K + \alpha_1(r)\partial_t K + \beta_0(r)\partial_t^2 H_1 + \beta_1(r)H_1,\\
\partial_r H_1 &= \gamma_0(r)\partial_t^3 K + \gamma_1(r)\partial_t K + \delta_0(r)\partial_t^2 H_1 + \delta_1(r)H_1,
\end{aligned} \tag{118}$$

[12]It can be shown that Eqs. (117) are consistent with the results of Zerilli [18] if one assumes the following time dependence for the modes:

$$\begin{aligned}
H_0(t,r) &= H_0(r)e^{-i\omega t},\\
H_1(t,r) &= H_1(r)e^{-i\omega t},\\
K(t,r) &= K(r)e^{-i\omega t}.
\end{aligned}$$

where the Greek-lettered functions are explicitly given by

$$
\begin{aligned}
\alpha_0(r) &= \left[\frac{f(r)}{2}\left(\frac{(\ell+2)(\ell-1)}{r}+\frac{6M}{r^2}\right)\right]^{-1}, \\
\alpha_1(r) &= -\frac{r-3M}{r^2 f(r)}+\left[\frac{(\ell+2)(\ell-1)}{r}+\frac{6M}{r^2}\right]^{-1}\left[\frac{(\ell+2)(\ell-1)}{r^2}+\frac{2M(r-3M)}{r^4 f(r)}\right], \\
\beta_0(r) &= -2\left[(\ell+2)(\ell-1)+\frac{6M}{r}\right]^{-1}, \\
\beta_1(r) &= -\frac{M}{r^5}\ell(\ell+1)\left[\frac{(\ell+2)(\ell-1)}{r^2}+\frac{6M}{r^3}\right]^{-1}+\frac{\ell(\ell+1)}{2r^2}, \\
\gamma_0(r) &= \frac{2}{f(r)^2}\left[\frac{(\ell+2)(\ell-1)}{r^2}+\frac{6M}{r^3}\right]^{-1}, \\
\gamma_1(r) &= \frac{1}{f(r)}\left(1+\left[\frac{(\ell+2)(\ell-1)}{r^2}+\frac{6M}{r^3}\right]^{-1}\left[\frac{(\ell+2)(\ell-1)}{r^2}+\frac{2M(r-3M)}{r^4 f(r)}\right]\right), \\
\delta_0(r) &= -\frac{2}{f(r)}\left[\frac{(\ell+2)(\ell-1)}{r}+\frac{6M}{r^2}\right]^{-1}, \\
\delta_1(r) &= -\frac{1}{f(r)}\left[\frac{\ell(\ell+1)}{r^4}+\frac{2M}{r^2}\right].
\end{aligned}
\tag{119}
$$

We introduce the new variable $P := \partial_t K$ and rewrite Eqs. (118) as

$$
\begin{aligned}
\partial_r P &= \left[\alpha_0(r)\partial_t^2+\alpha_1(r)\right]P+\left[\beta_0(r)\partial_t^2+\beta_1(r)\right]H_1, \\
\partial_r H_1 &= \left[\gamma_0(r)\partial_t^2+\gamma_1(r)\right]P+\left[\delta_0(r)\partial_t^2+\delta_1(r)\right]H_1.
\end{aligned}
\tag{120}
$$

Next, we define the transformation

$$
\begin{aligned}
P &= p(r)\hat{P}+q(r)\hat{H}_1, \\
H_1 &= v(r)\hat{P}+w(r)\hat{H}_1,
\end{aligned}
\tag{121}
$$

imposing that

$$
\frac{d\hat{P}}{dr^*}=\hat{H}_1, \qquad \frac{d\hat{H}_1}{dr^*}=\frac{d^2\hat{P}}{dr^{*2}}=\left[V(r^*)+\partial_t^2\right]\hat{P}, \qquad \frac{dr}{dr^*}:=f(r).
\tag{122}
$$

Here, r^* is the usual tortoise coordinate. The r-derivative of Eqs. (121) is

$$
\begin{aligned}
\partial_r P &= \frac{dp(r)}{dr}\hat{P}+p(r)\partial_r\hat{P}+\frac{dq(r)}{dr}\hat{H}_1+q(r)\partial_r\hat{H}_1 \\
&= \frac{dp(r)}{dr}\hat{P}+\frac{p(r)}{f(r)}\hat{H}_1+\frac{dq(r)}{dr}\hat{H}_1+\frac{q(r)}{f(r)}\left[V(r^*)+\partial_t^2\right]\hat{P}, \\
\partial_r H_1 &= \frac{dv(r)}{dr}\hat{P}+v(r)\partial_r\hat{P}+\frac{dw(r)}{dr}\hat{H}_1+w(r)\partial_r\hat{H}_1 \\
&= \frac{dv(r)}{dr}\hat{P}+\frac{v(r)}{f(r)}\hat{H}_1+\frac{dw(r)}{dr}\hat{H}_1+\frac{w(r)}{f(r)}\left[V(r^*)+\partial_t^2\right]\hat{P}.
\end{aligned}
\tag{123}
$$

Equating these two expressions to (120) and collect all terms that multiply $\partial_t^2\hat{P}$, $\hat{P}$ and $\hat{H}_1$ on the left-and right-hand-side yields a system of eight equations:

$$\begin{aligned}
\alpha_0(r)p(r) + \beta_0(r)v(r) &= \frac{1}{f(r)}q(r),\\
\alpha_1(r)p(r) + \beta_1(r)v(r) &= \frac{dp(r)}{dr} + \frac{1}{f(r)}q(r)V(r^*),\\
\alpha_0(r)q(r) + \beta_0(r)w(r) &= 0,\\
\alpha_1(r)q(r) + \beta_1(r)w(r) &= \frac{dq(r)}{dr} + \frac{1}{f(r)}p(r),\\
\gamma_0(r)p(r) + \delta_0(r)v(r) &= \frac{1}{f(r)}w(r),\\
\gamma_1(r)p(r) + \delta_1(r)v(r) &= \frac{dv(r)}{dr} + \frac{1}{f(r)}w(r)V(r^*),\\
\gamma_0(r)q(r) + \delta_0(r)w(r) &= 0,\\
\gamma_1(r)q(r) + \delta_1(r)w(r) &= \frac{dw(r)}{dr} + \frac{1}{f(r)}v(r).
\end{aligned} \tag{124}$$

This system can be solved for the functions $p(r)$, $q(r)$, $v(r)$, $w(r)$ and $V(r^*)$. The results are

$$p(r) = \frac{\lambda(\lambda+1)r^2 + 3M(\lambda r + 2M)}{r^2(\lambda r + 3M)}, \tag{125}$$

$$q(r) = 1, \tag{126}$$

$$v(r) = \frac{\lambda r^2 - 3\lambda M r - 3M^2}{rf(r)(\lambda r + 3M)}, \tag{127}$$

$$w(r) = \frac{r}{f(r)}, \tag{128}$$

$$V(r^*) = 2f(r)\frac{\lambda^2(\lambda+1)r^3 + 3\lambda^2 M r^2 + 9M^2(\lambda r + M)}{r^3(\lambda r + 3M)^2}, \tag{129}$$

where we defined

$$\lambda := \frac{(\ell+2)(\ell-1)}{2}. \tag{130}$$

By virtue of transformation (122), the variable $\hat{P}$ satisfies the Zerilli equation:

$$\frac{d^2\hat{P}}{dr^{*2}} = \left[V(r^*) + \partial_t^2\right]\hat{P}. \tag{131}$$

An expression for $\hat{P}$ is found by reversing Eqs. (121); we isolate $\hat{H}_1$ from the second equation and $\hat{P}$ from the first, insert the expression for $\hat{H}_1$ into the expression for $\hat{P}$ and obtain

$$\hat{P} = \left(p(r) - \frac{q(r)v(r)}{w(r)}\right)^{-1}\left(P - \frac{q(r)}{w(r)}H_1\right) \tag{132}$$

$$= \frac{r^2}{\lambda r + 3M}\left[\partial_t K - \frac{f(r)}{r}H_1\right]. \tag{133}$$

Clearly, this implies that in terms of $\hat{P}$, Eq. (131) is still a third-order differential equation in t. We prefer it in the form of a second-order equation. To this extent, we first substitute for H_1 from the first equation in (117), which gives

$$\hat{P} = \frac{r^2}{\lambda r + 3M}\left[\partial_t K - \frac{2r}{\ell(\ell+1)}\left(\frac{r-3M}{r^2}\partial_t K + f(r)\partial_t\partial_r K - \frac{f(r)}{r}\partial_t H_0\right)\right]. \tag{134}$$

Defining the Zerilli-Moncrief (ZM) function via

$$\hat{P} := \partial_t \Psi_{\mathrm{ZM}} \tag{135}$$

implies that also Ψ_{ZM} satisfies the Zerilli equation,

$$\frac{d^2\Psi_{\mathrm{ZM}}}{dr^{*2}} = \left[V_{\mathrm{zm}}(r^*) + \partial_t^2\right]\Psi_{\mathrm{ZM}}, \tag{136}$$

and (134) shows that Ψ_{ZM} is in this case given by

$$\Psi_{\mathrm{ZM}} = \frac{r^2}{\lambda r + 3M}\left[K - \frac{2r}{\ell(\ell+1)}\left(\frac{r-3M}{r^2}K + f(r)\partial_r K - \frac{f(r)}{r}H_0\right)\right] \tag{137}$$

$$= \frac{r}{(\lambda+1)(\lambda r + 3M)}\left[(\lambda r + 3M)K - r^2 f(r)\partial_r K + r f(r) H_2\right] \tag{138}$$

(note that $H_0 = H_2$). The corresponding potential is

$$V_{\mathrm{zm}}(r^*) := V(r^*) = 2f(r)\frac{\lambda^2(\lambda+1)r^3 + 3\lambda^2 M r^2 + 9M^2(\lambda r + M)}{r^3(\lambda r + 3M)^2}. \tag{139}$$

Transforming back from r^* to r, we see that we obtain the form of the Zerilli equation we were looking for:

$$(\Box - V_{\mathrm{ZM}})\,\Psi_{\mathrm{ZM}} = 0, \tag{140}$$

with

$$V_{\mathrm{ZM}}(r) := \frac{1}{f(r)}V_{\mathrm{zm}}(r^*). \tag{141}$$

2.8.2 Decoupling covariantly

We have seen that the even-parity system can be reduced to a single wave equation through a series of manipulations of the linearized vacuum Einstein equations in coordinates. We will show in this section that the same results are obtained when rewriting the system in its covariant form and decoupling it by introducing a specially tailored function, namely the covariant ZM function,

$$\Psi_{\mathrm{ZM}} := \frac{r}{\lambda+1}\left[K + \frac{v}{\Lambda}\right], \tag{142}$$

where

$$v := r^a r^b f_{ab} - r r^a \mathscr{D}_a K, \tag{143}$$

$$\Lambda(r) := \lambda + \frac{3M}{r}. \tag{144}$$

In order to follow the calculations in Martel's work [9], it is more convenient to use the perturbed Einstein tensors rather than the Ricci tensors[13]. Explicit expressions for the Einstein tensors are provided in Appendix D. The vacuum Einstein equations are obtained by equating each component of

[13] Although in vacuum it suffices to compute only the perturbed Ricci tensors for the vacuum Einstein equations, using the Einstein tensors offers the additional advantage of generality, as it allows for the inclusion of a source term. However, this is not our primary motivation; given that the covariant calculations involve many tedious and non-trivial manipulations, we aim to stay as close as possible to Martel's methodology, which necessitates working with the linearized Einstein tensors.

$\delta G_{\mu\nu}$ to zero and splitting off the spherical harmonics:

$$\begin{aligned}
0 = Q_{ab} = {}& \mathscr{D}_c \mathscr{D}_{(b} f^c_{a)} - \frac{1}{2} g_{ab} \mathscr{D}_c \mathscr{D}_d f^{cd} - \frac{1}{2} \mathscr{D}_a \mathscr{D}_b f^c_c - \frac{1}{2} \left(\Box f_{ab} - g_{ab} \Box f^c_c \right) \\
& + \frac{2}{r} r_c \left(\mathscr{D}_{(b} f^c_{a)} - g_{ab} \mathscr{D}_d f^{cd} \right) - \frac{r^c}{r} \left(\mathscr{D}_c f_{ab} - g_{ab} \mathscr{D}_c f^d_d \right) + \frac{\ell(\ell+1)}{2r^2} f_{ab} \\
& - \frac{1}{r^2} g_{ab} r^c r^d f_{cd} - \frac{1}{r} g_{ab} (\mathscr{D}_c r_d) f^{cd} - \frac{\ell(\ell+1)}{2r^2} g_{ab} f^c_c - \mathscr{D}_a \mathscr{D}_b K \\
& + g_{ab} \Box K - \frac{2}{r} r_{(a} \mathscr{D}_{b)} K + \frac{3}{r} g_{ab} r^c \mathscr{D}_c K - \frac{(\ell+2)(\ell-1)}{2r^2} g_{ab} K,
\end{aligned} \tag{145}$$

$$0 = Q_a = \mathscr{D}_b f^b_a - \mathscr{D}_a f^b_b + \frac{r_a}{r} f^b_b - \mathscr{D}_a K, \tag{146}$$

$$0 = Q^\flat = \Box f^a_a - \mathscr{D}_a \mathscr{D}_b f^{ab} - \frac{2}{r} r^b \mathscr{D}_a f^a_b + \frac{r^a}{r} \mathscr{D}_a f^b_b - \frac{\ell(\ell+1)}{2r^2} f^a_a + \frac{2}{r} r^a \mathscr{D}_a K + \Box K, \tag{147}$$

$$0 = Q^\sharp = f^a_a. \tag{148}$$

We retain the components of $Q_{\mu\nu}$, which in [9] indicate source terms, as labels on the equations (even though they vanish in our vacuum spacetime). Since from $Q_{AB} = 0$ two equations follow, they are labelled $Q^\flat$ and $Q^\sharp$. We also renamed Q_{aA} to Q_a as there is no capital index on the right-hand-side of Eq. (146).

Note that upon inserting the trace condition, Eq. (148), into Eq. (146) gives

$$\mathscr{D}_b f^b_a = \mathscr{D}_a K. \tag{149}$$

Substituting this and the trace condition into Eq. (147) shows that the latter is consistent with the other equations and therefore does not contain any additional information. The only extra information originating from $Q_{AB} = 0$ is therefore the trace condition, Eq. (148).

(145) can be slightly simplified by making use of Eq. (30) to rewrite the term

$$-\frac{1}{r} g_{ab} (\mathscr{D}_c r_d) f^{cd} = -\frac{M}{r^3} g_{ab} f^c_c. \tag{150}$$

Using this simplification and substituting the trace condition into Eqs. (145)-(147) results in the system

$$\begin{aligned}
0 = Q_{ab} = {}& \mathscr{D}_c \mathscr{D}_{(b} f^c_{a)} - \frac{1}{2} g_{ab} \mathscr{D}_d \mathscr{D}_c f^{cd} - \frac{1}{2} \Box f_{ab} + \frac{2}{r} r^c \left(\mathscr{D}_{(b} f_{a)c} - g_{ab} \mathscr{D}_d f^d_c \right) \\
& - \frac{r^c}{r} \mathscr{D}_c f_{ab} + \frac{\lambda+1}{r^2} f_{ab} - \frac{1}{r^2} g_{ab} r^c r^d f_{cd} - \mathscr{D}_a \mathscr{D}_b K - g_{ab} \Box K \\
& - \frac{2}{r} r_{(a} \mathscr{D}_{b)} K + \frac{3}{r} r^c g_{ab} \mathscr{D}_c K - \frac{\lambda}{r^2} g_{ab} K,
\end{aligned} \tag{151}$$

$$0 = Q_a = \mathscr{D}_b f^b_a - \mathscr{D}_a K, \tag{152}$$

$$0 = Q^\flat = -\mathscr{D}_a \mathscr{D}_b f^{ab} - \frac{2}{r} r_a \mathscr{D}_b f^{ab} + \Box K + \frac{2}{r} r^a \mathscr{D}_a K. \tag{153}$$

By performing clever manipulations, Eqs. (151)-(153) can be transformed into a set of three differential equations involving only K and the scalar field v. This system can then be decoupled by introducing the ZM function.

We start by taking the trace of Eq. (151). Making use of Eq. (152), we obtain

$$0 = Q^a_a := g^{ab} Q_{ab} = \Box K - \frac{2\lambda}{r^2} K - \frac{2}{r^2} v, \tag{154}$$

which is the first equation of our new system. The second and third equations we seek are found by forming clever combinations of Eq. (151). Martel does not provide an explanation for the choice of these combinations, but we assume he was granted divine insight. We start by forming the combination

$$f(r)Q^a_a - r^a r^b Q_{ab} = 0. \tag{155}$$

For the first term we simply substitute Eq. (154). The second term can be simplified right away by using a clever trick. We can linearize the Ricci tensor of the unperturbed background ($\mathcal{M}^2$) as

$$\begin{aligned}\delta\mathscr{R}_{ab} &= \delta\left(\frac{\mathscr{R}}{2}g_{ab}\right)\\ &= \frac{\delta\mathscr{R}}{2}g_{ab} + \frac{\mathscr{R}}{2}\delta g_{ab}\\ &= \frac{\delta\mathscr{R}}{2}g_{ab} + \frac{\mathscr{R}}{2}\gamma_{ab}\end{aligned} \tag{156}$$

where we used Eq. (23) in the first line and $\delta g_{ab} := \gamma_{ab}$ in the third. Inserting expressions (49) for the linearized Ricci tensor and (50) for the Ricci scalar in (156), expanding in spherical harmonics and using the trace condition yields

$$\mathscr{D}_c\mathscr{D}_{(b}f^c_{a)} - \frac{1}{2}\Box f_{ab} - \frac{1}{2}g_{ab}\mathscr{D}_c\mathscr{D}_d f^{cd} = \frac{\mathscr{R}}{2}f_{ab} = \frac{2M}{r^3}f_{ab}. \tag{157}$$

We use this expression to replace the first three terms in Eq. (145). The result is substituted in (155) and after tedious rewriting we obtain

$$\begin{aligned}0 = f(r)Q^a_a - r^a r^b Q_{ab} = &-\frac{r^a r^b}{r}(r^c\mathscr{D}_c f_{ab} - r\mathscr{D}_a\mathscr{D}_b K) - \frac{\lambda+2}{r^2}r^a r^b f_{ab}\\ &+ \frac{3}{r}f(r)r^a\mathscr{D}_a K - \frac{\lambda}{r^2}f(r)K.\end{aligned} \tag{158}$$

Further manipulation of this equation requires us to rewrite

$$\begin{aligned}r^a r^b r^c\mathscr{D}_c f_{ab} &= r^c\mathscr{D}_c\left(r^a r^b f_{ab}\right) - 2r^c r^a(\mathscr{D}_c r^b)f_{ab}\\ &= r^c\mathscr{D}_c(v + rr^a\mathscr{D}_a K) - \frac{2M}{r^2}(v + rr^a\mathscr{D}_a K)\\ &= r^a\mathscr{D}_a v - \frac{2M}{r^2}v + rr^a r^b\mathscr{D}_a\mathscr{D}_b K + \left(1 - \frac{3M}{r}\right)r^a\mathscr{D}_a K,\end{aligned} \tag{159}$$

where we used relation (30) and the definition of v to go from the first to the second line. Substituting Eq. (159) in Eq. (158) shows we have arrived at an equation only in terms of K and v,

$$0 = f(r)Q^a_a - r^a r^b Q_{ab} = -\frac{1}{r}r^a\mathscr{D}_a v - \frac{1}{r^2}\left(\lambda + 2 - \frac{2M}{r}\right)v - \frac{\Lambda}{r}r^a\mathscr{D}_a K - \frac{\lambda}{r^2}f(r)K. \tag{160}$$

This is the second equation of our new system.

Deriving the third equation requires most effort. It is obtained from the combination

$$\frac{2}{r}r^a r^b Q_{ab} + r^b\mathscr{D}_b Q^a_a = 0. \tag{161}$$

For the first term we use Eq. (151), while for the second term we use Eq. (154). A tedious algebraic exercise shows us that

$$\begin{aligned}\frac{2}{r}r^a r^b Q_{ab} + r^b\mathscr{D}_b Q^a_a = 0 = &\frac{2}{r}r^a r^b\mathscr{D}_c\mathscr{D}_a f^c_b - \frac{f(r)}{r}\mathscr{D}_a\mathscr{D}_b f^{ab} - \frac{1}{r}r^a r^b\Box f_{ab} - \frac{4}{r^2}f(r)r^a\mathscr{D}_b f^b_a\\ &+ \frac{2}{r^3}(\lambda + 2f(r))r^a r^b f_{ab} + \frac{2}{r}f(r)\Box K + r^a\mathscr{D}_a(\Box K)\\ &- \frac{2}{r^2}\left(\lambda - \frac{M}{r}\right)r^a\mathscr{D}_a K + \frac{2\lambda}{r^3}f(r)K.\end{aligned} \tag{162}$$

Eq. (162) can be further simplified by rewriting the first, third and seventh terms. For this we need the following results:

- The first term is rewritten by commuting the covariant derivatives and using Eq. (21) and Eq. (152):

$$\begin{aligned}\frac{2}{r}r^ar^b\mathscr{D}_c\mathscr{D}_af^c_b &= \frac{2}{r}r^ar^b\left(\mathscr{D}_a\mathscr{D}_cf^c_b + \mathscr{R}_{cab}{}^df^c_d + \mathscr{R}_{ca}{}^c{}_df^d_b\right)\\ &= \frac{2}{r}r^ar^b\mathscr{D}_a\mathscr{D}_bK + \frac{8M}{r^4}r^ar^bf_{ab}. \end{aligned} \tag{163}$$

- The third term is rewritten using

$$\begin{aligned} r^ar^b\Box f_{ab} &= r^ar^bg^{cd}\mathscr{D}_d\mathscr{D}_cf_{ab}\\ &= g^{cd}\mathscr{D}_d\left(r^ar^b\mathscr{D}_cf_{ab}\right) - g^{cd}(\mathscr{D}_dr^a)r^b\mathscr{D}_cf_{ab} - g^{cd}r^a(\mathscr{D}_dr^b)\mathscr{D}_cf_{ab}\\ &= g^{cd}\mathscr{D}_d\mathscr{D}_c\left(r^ar^bf_{ab}\right) - \mathscr{D}_d\left((\mathscr{D}^dr^a)r^bf_{ab}\right) - \mathscr{D}_d\left(r^a(\mathscr{D}^dr^b)f_{ab}\right)\\ &\quad - (\mathscr{D}^cr^a)r^b\mathscr{D}_cf_{ab} - r^a(\mathscr{D}^cr^b)\mathscr{D}_cf_{ab}\\ &= \Box(r^ar^bf_{ab}) - \mathscr{D}_c\left[((\mathscr{D}^cr^a)r^b + (\mathscr{D}^cr^b)r^a)f_{ab}\right] - 2(\mathscr{D}^cr^a)r^b\mathscr{D}_cf_{ab}\\ &= \Box(r^ar^bf_{ab}) - 2(\mathscr{D}_c\mathscr{D}^cr^a)r^bf_{ab} - 2(\mathscr{D}^cr^a)(\mathscr{D}_cr^b)f_{ab} - 4(\mathscr{D}^cr^a)r^b\mathscr{D}_cf_{ab}\\ &= \Box(r^ar^bf_{ab}) + \frac{4M}{r^3}r^ar^bf_{ab} - \frac{4M}{r^2}r^a\mathscr{D}_bf^b_a\\ &= \Box v + \frac{4M}{r^3}v + r\Box\left(r^a\mathscr{D}_aK\right) + 2r^ar^b\mathscr{D}_a\mathscr{D}_bK + \frac{4M}{r^2}r^a\mathscr{D}_aK, \end{aligned} \tag{164}$$

  where we used relation (30) in going from the third to the fourth line. In the last step we inserted the definition of v and made use of the fact that

$$\begin{aligned} \Box\left(rr^a\mathscr{D}_aK\right) &= \mathscr{D}^b\mathscr{D}_b(rr^a\mathscr{D}_aK)\\ &= r\mathscr{D}^b\mathscr{D}_b(r^a\mathscr{D}_aK) + r^b\mathscr{D}_b(r^a\mathscr{D}_aK) + \mathscr{D}^b(r_br^a\mathscr{D}_aK)\\ &= r\Box(r^a\mathscr{D}_aK) + r^b(\mathscr{D}_br^a)\mathscr{D}_aK + 2r^ar^b\mathscr{D}_a\mathscr{D}_bK + (\mathscr{D}^br_b)r^a\mathscr{D}_aK + r_b(\mathscr{D}^br^a)\mathscr{D}_aK\\ &= r\Box(r^a\mathscr{D}_aK) + 2r^ar^b\mathscr{D}_a\mathscr{D}_bK + \frac{4M}{r^2}r^a\mathscr{D}_aK. \end{aligned} \tag{165}$$

- Finally, the seventh term can be expressed as

$$\begin{aligned} r^a\mathscr{D}_a\left(\Box K\right) &= g^{bc}r^a\mathscr{D}_a\mathscr{D}_c\mathscr{D}_bK\\ &= g^{bc}r^a\left(\mathscr{D}_c\mathscr{D}_a\mathscr{D}_bK + \mathscr{R}_{acb}{}^d\mathscr{D}_dK\right)\\ &= g^{bc}\mathscr{D}_c\mathscr{D}_b\left(r^a\mathscr{D}_aK\right) - g^{bc}\mathscr{D}_c\left((\mathscr{D}_br^a)\mathscr{D}_aK\right) - g^{bc}(\mathscr{D}_cr^a)\mathscr{D}_a\mathscr{D}_bK - \frac{2M}{r^3}r^a\mathscr{D}_aK\\ &= \Box\left(r^a\mathscr{D}_aK\right) - 2(\mathscr{D}^br^a)\mathscr{D}_a\mathscr{D}_bK\\ &= \Box\left(r^a\mathscr{D}_aK\right) - \frac{2M}{r^2}\Box K, \end{aligned} \tag{166}$$

  where we again used relation (30) in the last step.

Substituting these results into (162) and using the definition of v, we obtain the third equation of our new system:

$$\begin{aligned} 0 = \frac{2}{r}r^ar^bQ_{ab} + r^b\mathscr{D}_bQ^a_a = &-\frac{1}{r}\Box v + \frac{2}{r^3}\left[\lambda + 2\left(1 - \frac{M}{r}\right)\right]v + \frac{1}{r}\left(1 - \frac{4M}{r}\right)\Box K\\ &+ \frac{6M}{r^3}r^a\mathscr{D}_aK + \frac{2\lambda}{r^3}f(r)K. \end{aligned} \tag{167}$$

The perturbation equations, (145)-(148), have now been reformulated as a system of three equations in terms of the two variables K and v:

$$\begin{aligned}
0 &= \Box K - \frac{2\lambda}{r^2}K - \frac{2}{r^2}v, \\
0 &= -\frac{1}{r}r^a \mathscr{D}_a v - \frac{1}{r^2}\left(\lambda + 2 - \frac{2M}{r}\right)v - \frac{\Lambda}{r}r^a\mathscr{D}_a K - \frac{\lambda}{r^2}f(r)K, \\
0 &= -\frac{1}{r}\Box v + \frac{2}{r^3}\left[\lambda + 2\left(1 - \frac{M}{r}\right)\right]v + \frac{1}{r}\left(1 - \frac{4M}{r}\right)\Box K + \frac{6M}{r^3}r^a\mathscr{D}_a K + \frac{2\lambda}{r^3}f(r)K.
\end{aligned} \tag{168}$$

We eliminate[14] v in favour of Ψ_{ZM}:

$$\begin{aligned}
0 &= \Box K + \frac{6M}{r^3}K - \frac{2\Lambda(\lambda+1)}{r^3}\Psi_{\text{ZM}}, \\
0 &= -\frac{\Lambda(\lambda+1)}{r^2}r^a\mathscr{D}_a\Psi_{\text{ZM}} - \frac{\lambda+1}{r^3}\left[\lambda(\lambda+1) + \frac{3M}{r}\lambda + \frac{6M^2}{r^2}\right]\Psi_{\text{ZM}} + \frac{\Lambda(\lambda+1)}{r^2}K, \\
0 &= -\frac{\Lambda(\lambda+1)}{r^2}\Box\Psi_{\text{ZM}} + \frac{2(\lambda+1)}{r^3}\left(\lambda + \frac{6M}{r}\right)r^a\mathscr{D}_a\Psi_{\text{ZM}} \\
&\quad + \frac{2(\lambda+1)}{r^4}\left[\lambda(\lambda+1) + \frac{M}{r}(4\lambda - 3) + 18\frac{M^2}{r^2}\right]\Psi_{\text{ZM}} \\
&\quad + \frac{1}{r}\left(\lambda + 1 - \frac{M}{r}\right)\Box K - \frac{2}{r^3}\left(\lambda(\lambda+1) + \frac{3M}{r}(\lambda+1) + \frac{3M^2}{r^2}\right)K.
\end{aligned} \tag{169}$$

These three equations can, miraculously, be rewritten into a single wave equation by isolating $\Box K$ from the first equation and K from the second, and substituting both results into the third. The resulting equation is the Zerilli equation,

$$(\Box - V_{\text{ZM}})\,\Psi_{\text{ZM}} = 0, \tag{170}$$

where

$$V_{\text{ZM}}(r) = \frac{1}{r^2\Lambda^2}\left[2\lambda^2(1+\Lambda) + \frac{18M^2}{r^2}\left(\lambda + \frac{M}{r}\right)\right]. \tag{171}$$

This potential exactly matches the one we derived in coordinates (but is written in a slightly different form).

2.9 The Chandrasekhar/Darboux transformation

In Section 2.7 and 2.8, we have demonstrated a remarkable feature of the Schwarzschild spacetime: the complex systems of perturbation equations can be decoupled using a single function for each parity, namely Ψ_{RW} and Ψ_{ZM}, and written as two independent wave equations. It now remains to show that the RW equation and the Zerilli equation are related by a specific type of transformation.

This relationship is commonly referred to as the Chandrasekhar transformation, named after its discoverer. In his work [19], Chandrasekhar however failed to realize that the transformation he derived is actually a specific instance of the more general concept of a *Darboux transformation (DT)*. A DT is a broader notion that is applied to many areas of physics, allowing one to relate the solutions of second-order ordinary differential equations that are written in canonical form (that is, without first-order derivatives). In the context of black hole perturbation theory, the very existence of a DT between the odd- and even-parity master equations implies that, in such a spacetime, even and odd perturbations are isospectral [23].

In the following subsection we show explicitly that the RW and Zerilli equations can be related by a DT. Subsequently, in subsection 2.9.2, we demonstrate that the existence of a DT directly implies

[14] We have the freedom to eliminate either v or K from the system, but we choose to remove v.

isospectrality. This section follows the approach outlined in the work of Glampedakis et al. [23] and Chandrasekhar and Detweiler [22].

2.9.1 The relation between the RW and Zerilli equations

DTs in general relate two differential equations of the form

$$\begin{aligned} y''(x) + [\alpha - V_1(x)]y(x) = 0, \\ Y''(x) + [\alpha - V_2(x)]Y(x) = 0, \end{aligned} \tag{172}$$

with eigenvalue α, potentials $V_1(x)$ and $V_2(x)$, and the prime indicating differentiation with respect to the variable x. The DT between these equations is given by the linear relation

$$Y = y' + a(x)y, \tag{173}$$

connecting their solutions. Differentiating Eq. (173) twice with respect to x and substituting Eqs. (172) and (173) for y' and y'' yields

$$Y'' = (\alpha - 2a' - V_1)Y + (a^2 - a' + \alpha - V_1)'y = 0. \tag{174}$$

Equating this to the second differential equation in (172) shows that we obtain two constraints,

$$V_2 = 2a' + V_1, \tag{175}$$

$$(a^2 - a' + \alpha - V_1)' = 0 \rightarrow a^2 - a' + \alpha - V_1 = C, \tag{176}$$

with C a constant. Combining these constraints yields two equations that completely determine the transformation:

$$a = \frac{(2\alpha - V_1 - V_2)'}{2(V_1 - V_2)}, \tag{177}$$

$$a' = -\frac{1}{2}(V_1 - V_2). \tag{178}$$

In the context of this thesis, the analogues of (172) are the Schrödinger-like form of the RW and Zerilli equations, which we will reiterate:

$$\frac{d^2\Psi_{\rm RW}}{dr^{*2}} + \left[\omega^2 - V_{\rm rw}(r^*)\right]\Psi_{\rm RW} = 0, \tag{179}$$

$$\frac{d^2\Psi_{\rm ZM}}{dr^{*2}} + \left[\omega^2 - V_{\rm zm}(r^*)\right]\Psi_{\rm ZM} = 0. \tag{180}$$

We identify the eigenvalue $\alpha = \omega^2$, $V_1 = V_{\rm rw}$ and $V_2 = V_{\rm zm}$. Moreover, the variable $x = r^*$, such that $\partial_x = f(r)\partial_r$. It is then straightforward to verify from Eq. (177) that

$$a(r) = \frac{f(r)\partial_r(2\omega^2 - V_{\rm rw} - V_{\rm zm})}{2(V_{\rm rw} - V_{\rm zm})} = \frac{\lambda(\lambda+1)}{3M} + \frac{3Mf(r)}{r(\lambda r + 3M)}. \tag{181}$$

The second constraint, Eq. (178), implies that the RW and Zerilli potentials are related by

$$V_{\rm zm} = V_{\rm rw} + 2a', \tag{182}$$

which is readily verified by inserting expressions (98) and (139) for the potentials. A proof that this DT agrees with Chandrasekhar's transformation is given in [23].

2.9.2 Isospectrality of the Darboux transformation

In this section we will show that the existence of a DT between the potentials of the RW and Zerilli equations implies that their QNM frequencies are isospectral. We make use of the fact that DTs in general have the inherent property of preserving the transmission and reflection amplitudes of the potentials they relate.

Transmission amplitudes describe how an incoming plane wave propagates through the potential, while reflection amplitudes determine how the wave is scattered back. Together, these quantities fully characterize the asymptotic properties of the wave. QNMs are defined precisely by this asymptotic behaviour[15]; they correspond to the values of ω for which the wave is purely outgoing at $r^* = +\infty$ and purely ingoing at $r^* = -\infty$. If the transmission and reflection coefficients of two potentials are equal, their QNM spectra must be identical, which means that they are isospectral. We will therefore explicitly demonstrate the isospectrality of Darboux-related potentials by proving the equality of their transmission and reflection coefficients.

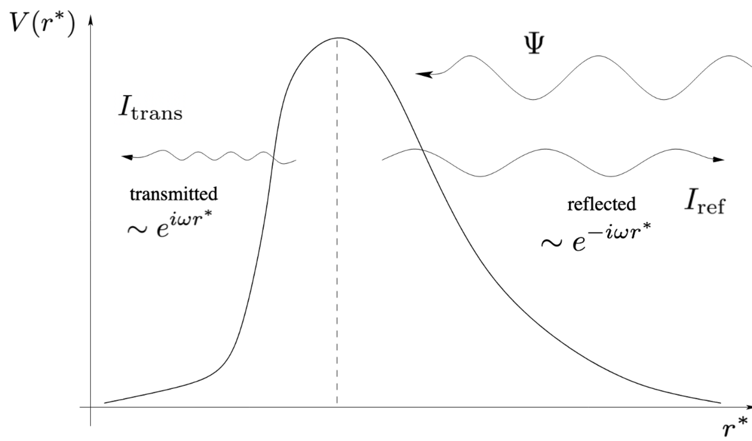

Figure 2: A plane wave impinging on the RW or Zerilli potential is partially reflected and partially transmitted. Adapted from: [44].

Characteristics that are key to the argument are non-singular, barrier-like and short-ranged[16] nature of the RW and Zerilli potentials. The latter implies that the asymptotic behaviour of the solutions Ψ_{RW} and Ψ_{ZM} is in the form of plane waves,

$$\Psi \sim e^{\pm i\omega r^*} \qquad (r^* \to \pm\infty), \tag{183}$$

which follows from Eq. (97) and Eq. (136) when their respective potentials vanish. Let Ψ_{RW} describe a wave coming from $r^* = +\infty$ and hitting the potential $V_{\text{rw}}(r^*)$. The wave will be partially reflected and partially transmitted, which is described by the solution

$$\begin{aligned} \Psi_{\text{RW}}(+\infty) =& A_{\text{in}}(\omega)e^{-i\omega r^*} + A_{\text{out}}(\omega)e^{i\omega r^*}, \\ \Psi_{\text{RW}}(-\infty) =& B_{\text{in}}(\omega)e^{-i\omega r^*}. \end{aligned} \tag{184}$$

The accompanying reflection and transmission coefficients are given by

$$I_{\text{ref}} = \frac{|A_{\text{out}}|^2}{|A_{\text{in}}|^2}, \qquad I_{\text{trans}} = \frac{|B_{\text{in}}|^2}{|A_{\text{in}}|^2}. \tag{185}$$

[15] At this point it is particularly convenient to work with the tortoise coordinate, as $r^* \to r$ for $r \to +\infty$ and $r^* \to -\infty$ for $r \to 2M$, the latter corresponding to the event horizon of the black hole. Hence, in terms of the tortoise coordinate, there is no coordinate singularity at the event horizon [44].

[16] With short-ranged, we mean they must decay faster than $(r^*)^{-1}$ as $r^* \to \pm\infty$.

Since the QNM frequencies are the values of ω for which the wave is purely outgoing at $r^* = +\infty$ and purely ingoing at $r^* = -\infty$, we must have

$$A_{\text{in}}(\omega) \stackrel{!}{=} 0, \tag{186}$$

which means that $I_{\text{ref}}^{-1} = I_{\text{trans}}^{-1} = 0$ for QNMs.

Inserting Eqs. (184) into the linear Darboux relation (173) provides the form of $\Psi_{\text{ZM}}(\pm\infty)$:

$$\begin{aligned}
\Psi_{\text{ZM}}(+\infty) =&\partial_{r^*}\Psi_{\text{RW}}(+\infty) + a(+\infty)\Psi_{\text{RW}}(+\infty) \\
=&[-i\omega + a(+\infty)]A_{\text{in}}(\omega)e^{-i\omega r^*} + [i\omega + a(+\infty)]A_{\text{out}}(\omega)e^{i\omega r^*} \\
:=&\mathcal{A}_{\text{in}}e^{-i\omega r^*} + \mathcal{A}_{\text{out}}e^{i\omega r^*}, \\
\Psi_{\text{ZM}}(-\infty) =&\partial_{r^*}\Psi_{\text{RW}}(-\infty) + a(-\infty)\Psi_{\text{RW}}(-\infty) \\
=&[-i\omega + a(-\infty)]B_{\text{in}}(\omega)e^{-i\omega r^*} \\
:=&\mathcal{B}_{\text{in}}(\omega)e^{-i\omega r^*}.
\end{aligned} \tag{187}$$

A Taylor expansion of (181) around $r^* \to \pm\infty$ gives

$$a(\pm\infty) = a_0 + \mathcal{O}\left(\frac{1}{r^*}\right), \qquad a_0 = \frac{\lambda(\lambda+1)}{3M}, \tag{188}$$

such that the reflection and transmission coefficients are

$$\begin{aligned}
\mathcal{I}_{\text{ref}} &= \frac{|\mathcal{A}_{\text{out}}|^2}{|\mathcal{A}_{\text{in}}|^2} = I_{\text{ref}}\frac{|i\omega + a_0|^2}{|i\omega - a_0|^2} = I_{\text{ref}}, \\
\mathcal{I}_{\text{trans}} &= \frac{|\mathcal{B}_{\text{in}}|^2}{|\mathcal{A}_{\text{in}}|^2} = I_{\text{trans}}\frac{|i\omega - a_0|^2}{|i\omega - a_0|^2} = I_{\text{trans}}.
\end{aligned} \tag{189}$$

The even QNMs satisfy the similar condition

$$\mathcal{A}_{\text{in}}(\omega) \stackrel{!}{=} 0 \longrightarrow \mathcal{I}_{\text{ref}}^{-1} = \mathcal{I}_{\text{trans}}^{-1} = 0. \tag{190}$$

The fact that $\mathcal{I}_{\text{trans}} = I_{\text{trans}}$ and $\mathcal{I}_{\text{ref}} = I_{\text{ref}}$ for the Darboux-related potentials V_{zm} and V_{rw}, and the fact that they satisfy the same QNM condition proves that the QNM spectra of both potentials coincide. This proves that the RW and Zerilli equations are isospectral.

3 Metric Perturbations of the Black String Spacetime

In Section 2, we covered the formalism of metric perturbations for the Schwarzschild black hole, and established the isospectrality of its QNM frequencies. We now apply this approach to the five-dimensional spacetime of a black string, aiming to investigate whether a similar isospectral relationship holds in this higher-dimensional setting.

When considering black holes in five dimensions, a natural starting point is the Schwarzschild spacetime, which can be extended by an additional dimension in (at least) two distinct ways. The first option is to introduce an extra angular dimension, resulting in a hyper-spherically symmetric black hole. Alternatively, we can extend the spacetime uniformly along a fourth spatial dimension, forming a black string [45]. If we would extend our spacetime even further in this manner, black strings can be viewed as lower-dimensional cases of the more general *black p-brane*, a class of black objects predicted by ST. A p-brane is effectively the p-dimensional counterpart of a black hole. [46]. A black string therefore represents a p = 5 -dimensional generalization of a black hole.

The structure of this section is as follows. First, we briefly examine a key property of the black string, namely its instability. In Section 3.2, we discuss the form of the metric describing a black string. Following this, we apply the perturbation formalism to the black string spacetime, aiming to derive the five-dimensional Regge-Wheeler and Zerilli equations in Sections 3.3 - 3.8. Section 3.9 covers a short discussion on the number of variables in our results.

3.1 Instability

Schwarzschild black holes have been demonstrated to be stable under linear metric perturbations by Regge and Wheeler [16]. In a study of higher-dimensional black holes, Gregory and Laflamme [47] established that black strings and p-branes are in fact unstable when subjected to such perturbations, an effect now referred to as the Gregory-Laflamme instability. This instability is physical, which suggests that the event horizon may undergo fragmentation into several higher-dimensional analogues of spherical black holes.

Since linear perturbation theory cannot predict the final state of black strings, more recent studies have sought to determine it numerically using nonlinear methods (see Ref. [48–50]). These studies have revealed that sufficiently thin black strings, when perturbed, evolve into a sequence of three-dimensional spherical black holes of varying sizes, connected by black string segments (of comparable radius). Each of these local string segments is itself unstable, driving a self-similar cascade down to arbitrarily small scales[17]. However, a definitive consensus on whether this constitutes the true final state of the black string has yet to be reached.

The Gregory-Laflamme instability is relevant for our discussion of perturbations of a black string because it imposes a limiting condition on its length. In the next section, we will discuss the concrete implications of this limitation.

3.2 Metric

The metric of a black string is obtained by introducing an additional uniform spatial dimension to the Schwarzschild metric, resulting in a spacetime with cylindrical symmetry. The uniformity of this extra dimension implies that it is independent of any of the coordinates. Consequently, the line element for this spacetime is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2 + dz^2. \tag{191}$$

[17] Notably, this bifurcation could lead to the formation of a naked singularity, potentially violating the cosmic censorship hypothesis. For a discussion, see Ref. [49, 51, 52].

A visualization of the event horizon of this spacetime is given in Figure 3. This metric evidently exhibits translational symmetry along the z-direction, suggesting that the black string can, in principle, extend indefinitely. There are two simple heuristic arguments why this is physically impossible. First, an infinitely long string would possess an infinite total mass, which is clearly unphysical. Second, the string's length is fundamentally limited by the Gregory-Laflamme instability. As discussed in Section 3.1, this instability causes the string to fragment when its length exceeds a critical value (L_c). Fragmentation results from the growing modes of the instability if this threshold is exceeded [48]. To circumvent these issues, we assume the black string is compact, meaning it has a finite length L, satisfying $L_c \gg L \gg R_s$, where R_s is string's radius. This assumption not only ensures a finite total mass but also aligns the black string with the string-theoretical (Kaluza-Klein) framework.

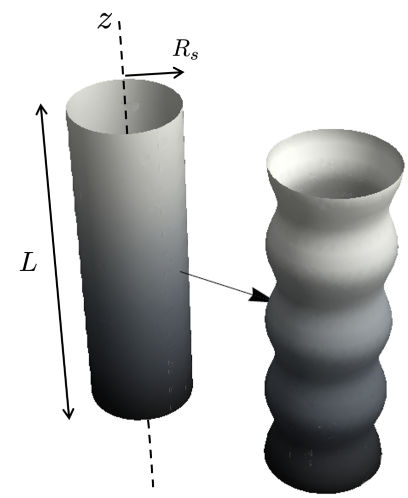

Figure 3: A visualization of the event horizon of a black string (left), with the effect of the instability caused by a perturbation (right). Adapted from [1].

3.3 Perturbing the black string

We will now perturb the metric of the black string, similarly to how we perturbed the Schwarzschild metric in Section 3. As before, we separate the full manifold into two submanifolds. We naturally choose[18] to incorporate the z-dimension into $\mathcal{M}^3$. The line element then takes the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + dz^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) := g_{ab}dx^a dx^b + r^2\Omega_{AB}dx^A dx^B. \tag{192}$$

with the lowercase Latin indices indicating

$$x^a = (t, r, z), \qquad a = 0, 1, 2, \tag{193}$$

while the capital Latin indices still take on either θ or ϕ:

$$x^A = (\theta, \phi), \qquad A = 3, 4. \tag{194}$$

[18] In principle we could integrate the extra dimension into $\mathcal{S}^3$, but this would introduce unnecessary complexity. We prefer to maintain the advantage that the angular part of the perturbations naturally separates, something we used to our advantage in the four-dimensional case. It is important to note that absorbing the uniform z-dimension into the angular coordinates differs from introducing a new *angular* dimension, as the latter would lead to hyperspherical symmetry rather than cylindrical symmetry.

Curvature quantities will inevitably change upon adding the extra dimension. We expect however that adding a uniform dimension does not alter the constant-curvature nature of the submanifold[19], allowing us to use Eq. (18) for $d = 3$,

$$\mathscr{R}_{abcd} = \frac{2M}{3r^3}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{195}$$

where we used the fact that the Ricci scalar of $\mathcal{M}^3$ is equal to that of $\mathcal{M}^2$. It is not surprising that the Riemann tensor differs from its two-dimensional counterpart, as different spacetimes inherently come with distinct background curvatures.

Since the extra dimension does not depend on any of the coordinates, derivatives of the metric remain unchanged, and the Christoffel symbols are identical to those in the four-dimensional case (after all, Christoffel symbols with one or more z-indices simply vanish). Consequently, the linearized connection (Eq. (37)), along with the Ricci tensor (Eq. (49)) and Einstein tensor (Eq. (51)) , are also unaffected. We verify this by evaluating them in five dimensions using the script 5D_PERT_COORDINATES.NB. In the next section, we will observe that differences with perturbations of the Schwarzschild spacetime emerge when we apply the spherical harmonics decomposition.

3.4 Decomposition into spherical harmonics

The perturbed metric components are again decomposed in scalar, vector and tensor harmonics of even and odd parity as

$$\begin{aligned}
\gamma_{ab} &= \sum_{l,m} f_{ab}^{\ell m}(t,r,z)Y^{\ell m}, \\
\gamma_{aA} &= \sum_{l,m} \left\{ j_a^{\ell m}(t,r,z)Y_A^{\ell m} + h_a^{\ell m}(t,r,z)X_A^{\ell m} \right\}, \\
\gamma_{AB} &= \sum_{l,m} \left\{ r^2 K^{\ell m}(t,r,z)\Omega_{AB}Y^{\ell m} + r^2 G^{\ell m}(t,r,z)Y_{AB}^{\ell m} + h_3^{\ell m}(t,r,z)X_{AB}^{\ell m} \right\}.
\end{aligned} \tag{196}$$

The addition of an extra coordinate implies that the mode functions take the form

$$f_{ab}^{\ell m} = \begin{pmatrix} f(r)H_0^{\ell m}(t,r,z) & H_1^{\ell m}(t,r,z) & H_3^{\ell m}(t,r,z) \\ H_1^{\ell m}(t,r,z) & \frac{1}{f(r)}H_2^{\ell m}(t,r,z) & H_4^{\ell m}(t,r,z) \\ H_3^{\ell m}(t,r,z) & H_4^{\ell m}(t,r,z) & H_5^{\ell m}(t,r,z) \end{pmatrix}, \tag{197}$$

$$j_a^{\ell m} = \begin{pmatrix} j_0^{\ell m}(t,r,z) \\ j_1^{\ell m}(t,r,z) \\ j_2^{\ell m}(t,r,z) \end{pmatrix}, \tag{198}$$

$$h_a^{\ell m} = \begin{pmatrix} h_0^{\ell m}(t,r,z) \\ h_1^{\ell m}(t,r,z) \\ h_2^{\ell m}(t,r,z) \end{pmatrix}. \tag{199}$$

Note that the mode $h_2^{\ell m}$ is now part of $h_a^{\ell m}$, while $h_3^{\ell m}$ is the new name for the odd-parity mode belonging to the tensor harmonic $X_{AB}^{\ell m}$. It is evident that there are 11 even-parity modes ($H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$, $H_3^{\ell m}$, $H_4^{\ell m}$, $H_5^{\ell m}$ $j_0^{\ell m}$, $j_1^{\ell m}$, $j_2^{\ell m}$, $K^{\ell m}$ and $G^{\ell m}$) and four odd-parity modes ($h_0^{\ell m}$, $h_1^{\ell m}$, $h_2^{\ell m}$ and $h_3^{\ell m}$).

[19]The assumption that $\mathcal{M}^3$ has constant curvature turns out to be incorrect. See the addendum at the end of this thesis

3.5 Gauge transformations

In total there are 15 variables ($H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$, $H_3^{\ell m}$, $H_4^{\ell m}$, $H_5^{\ell m}$ $j_0^{\ell m}$, $j_1^{\ell m}$, $j_2^{\ell m}$, $K^{\ell m}$, $G^{\ell m}$, $h_0^{\ell m}$, $h_1^{\ell m}$, $h_2^{\ell m}$ and $h_3^{\ell m}$) in the 11 even-parity and four odd-parity vacuum Einstein equations. We can again eliminate some of them by applying appropriate gauge transformations. Dividing the gauge vector $\Xi_\mu = (\Xi_a, \Xi_A)$ into vectors with even and odd parity,

$$\begin{aligned} \Xi_a &= \sum_{\ell m} \xi_a^{\ell m} Y^{\ell m}, \\ \Xi_A &= \sum_{\ell m} \left\{ \xi_3^{\ell m} Y_A^{\ell m} + \xi_4^{\ell m} X_A^{\ell m} \right\}, \end{aligned} \tag{200}$$

shows that there are four even-parity functions, $\xi_0^{\ell m}$, $\xi_1^{\ell m}$, $\xi_2^{\ell m}$ and $\xi_3^{\ell m}$, and one odd-parity function, $\xi_4^{\ell m}$. By exploiting the gauge freedom, we can therefore eliminate *four*[20] even-parity mode components (as opposed to three in the Schwarschild case) and one odd-parity mode component of the metric perturbations.

3.5.1 Odd-parity gauge transformations

The odd-parity gauge transformations are generated by $\Xi_\mu^{(\mathrm{odd})} = (0, \Xi_A^{(\mathrm{odd})})$. Analogously to the four-dimensional case, we find that the mode functions transform as

$$h_a^{\ell m} \longrightarrow h_a'^{\ell m} = h_a^{\ell m} - \mathscr{D}_a \xi_4^{\ell m} + \frac{2}{r} r_a \xi_4^{\ell m}, \tag{201}$$

$$h_3^{\ell m} \longrightarrow h_3'^{\ell m} = h_3^{\ell m} - 2\xi_4^{\ell m}. \tag{202}$$

We now find three gauge-invariant quantities:

$$\tilde{h}_a^{\ell m} = h_a^{\ell m} - \frac{1}{2}\partial_a h_3^{\ell m} + \frac{1}{r} r_a h_3^{\ell m}. \tag{203}$$

Eq. (202) shows that we can set $h_3^{\ell m} = 0$ by imposing that $\xi_4^{\ell m} = \frac{1}{2} h_3^{\ell m}$. This is the five-dimensional analogue of the RW gauge. Imposing $h_3^{\ell m} = 0$ implies

$$\tilde{h}_0^{\ell m} = h_0^{\ell m}, \tag{204}$$

$$\tilde{h}_1^{\ell m} = h_1^{\ell m}, \tag{205}$$

$$\tilde{h}_2^{\ell m} = h_2^{\ell m}. \tag{206}$$

3.5.2 Even-parity gauge transformations

The even-parity gauge transformations are generated by $\Xi_\mu^{(\mathrm{even})} = (\Xi_a^{(\mathrm{even})}, \Xi_A^{(\mathrm{even})})$, such that the mode functions transform as

$$f_{ab}^{\ell m} \longrightarrow f_{ab}'^{\ell m} = f_{ab}^{\ell m} - 2\mathscr{D}_{(a}\xi_{b)}^{\ell m}, \tag{207}$$

$$j_a^{\ell m} \longrightarrow j_a'^{\ell m} = j_a^{\ell m} - \xi_a^{\ell m} - \mathscr{D}_a \xi_3^{\ell m} + \frac{2}{r} r_a \xi_3^{\ell m}, \tag{208}$$

$$K^{\ell m} \longrightarrow K'^{\ell m} = K^{\ell m} + \frac{\ell(\ell+1)}{r^2}\xi_3^{\ell m} - \frac{2}{r} r^a \xi_a^{\ell m}, \tag{209}$$

$$G^{\ell m} \longrightarrow G'^{\ell m} = G^{\ell m} - \frac{2}{r^2}\xi_3^{\ell m}. \tag{210}$$

[20] The fact that we can fix an extra even-parity quantity compared to four dimensions is simply due to the fact that $j_a^{\ell m}$ is extended with one variable, $j_2^{\ell m}$.

The gauge-invariant quantities are

$$\tilde{f}_{ab}^{\ell m} = f_{ab}^{\ell m} - \mathscr{D}_{(a}\left(j_{b)}^{\ell m} - \frac{r^2}{2}\mathscr{D}_{b)}G^{\ell m}\right), \tag{211}$$

$$\tilde{K}^{\ell m} = K^{\ell m} + \frac{\ell(\ell+1)}{2}G^{\ell m} - \frac{2}{r}r^a\left(j_a^{\ell m} - \frac{r^2}{2}\mathscr{D}_a G^{\ell m}\right). \tag{212}$$

We can set $j_a^{\ell m} = 0$ and $G^{\ell m} = 0$ by choosing $\xi_3^{\ell m} = \frac{r^2}{2}G^{\ell m}$ and $\xi_a^{\ell m} = j_a^{\ell m} - \frac{r^2}{2}\mathscr{D}_a G^{\ell m}$ in the RW gauge. Imposing $j_a^{\ell m} = 0$ and $G^{\ell m} = 0$ implies that

$$\tilde{f}_{ab}^{\ell m} = f_{ab}^{\ell m}, \tag{213}$$

$$\tilde{K}^{\ell m} = K^{\ell m}. \tag{214}$$

This means that also in five dimensions, the perturbed quantities in the RW gauge are equal to their gauge-invariant counterparts. From now one we will pick up the practice of dropping the summation over ℓ and m.

3.6 Tools for checking calculations

The introduction of the extra dimension adds several variables and equations, making it considerably more difficult to decouple the perturbation equations. As a result, the calculations in the rest of this thesis will be lengthy and tedious. It is crucial to carefully track the information contained within the system of perturbation equations. We have three "tools" that aid us in assessing the correctness of our calculations:

- **Schwarzschild limit**: By setting all quantities related to the fifth dimension to zero (i.e. the new variables and z-derivatives), we can verify that we recover the four-dimensional results. To facilitate a clear comparison, it is helpful to work in terms of the dimension $d = 2, 3$, where the Schwarschild limit corresponds to $d = 2$, and the black string case corresponds to $d = 3$.
- **Counting variables and equations**: After manipulating the equations, we can count the total number of variables and the corresponding equations needed to describe them. This allows us to track the information in the system, ensuring it is not underdetermined.
- **Dimensional analysis**: It is always possible to check whether the terms in our calculations have the correct dimensions. For instance, the (linearized) Ricci tensor has dimensions $[\delta R_{\mu\nu}] = [R_{\mu\nu}] = m^{-2}$, since it contains second derivatives of the metric tensor. Note that throughout this thesis, we use natural units. To restore units, we must replace $M \to G_N M/c^2$ and $t \to ct$, with G_N Newton's constant and c the speed of light.

While none of these tools do not provide *proof* that we have obtained the correct results, they do minimize the risk of structural errors.

3.7 Regge-Wheeler equation

In this section, we aim to derive the RW equation in five dimensions. We start from the odd parity perturbation equations[21] in Appendix D for $d = 3$. We must be careful that this time,

$$-\frac{1}{r}h_b\mathscr{D}_a\mathscr{D}^b r = -\frac{1}{r}\frac{M}{r^2}\begin{pmatrix}h_0\\h_1\\0\end{pmatrix} \neq -\frac{M}{r^3}h_a = -\frac{1}{r}\frac{M}{r^2}\begin{pmatrix}h_0\\h_1\\h_2\end{pmatrix}, \tag{216}$$

impeding us from using the same simplification for $\delta R^{(\text{odd})}_{aB}$ as in four dimensions. We obtain the following vacuum Einstein equations:

$$\begin{aligned}
\delta R^{(\text{odd})}_{ab} &= 0,\\
\delta R^{(\text{odd})}_{aB} &= 0 = \Big[-\hat{\Box}h_a + \mathscr{D}_a\mathscr{D}^b h_b + \frac{4M}{3r^3}h_a - \frac{2}{r}r_a\mathscr{D}^b h_b + \frac{2}{r}r^b\mathscr{D}_a h_b - \frac{2}{r^2}r_a r^b h_b\\
&\qquad - \frac{2}{r}h_b\mathscr{D}_a\mathscr{D}^b r + \frac{\ell(\ell+1)}{r^2}h_a\Big]X_B,\\
\delta R^{(\text{odd})}_{AB} &= 0 = [\mathscr{D}^a h_a]X_{AB}.
\end{aligned} \tag{217}$$

3.7.1 Decoupling in coordinates

Eqs. (217) are evaluated in coordinates using the script 5D_PERT_COORDINATES.NB in terms of the dimension d. The results are

$$\begin{aligned}
\delta R^{(\text{odd})}_{tA} &= 0 = \frac{1}{2}\Bigg[f(r)\left(-\partial_r^2 h_0 + \left(\partial_r + \frac{2}{r}\right)\partial_t h_1\right) + \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\left(\frac{2}{d} - 3\right)\right)h_0\\
&\qquad + \partial_t\partial_z h_2 - \partial_z^2 h_0\Bigg]X_A,\\
\delta R^{(\text{odd})}_{rA} &= 0 = \frac{1}{2}\Bigg[-\frac{1}{f(r)}\left(\partial_r - \frac{2}{r}\right)\partial_t h_0 + \left(\frac{(\ell-1)(\ell+2)}{r^2} + \frac{2M}{r^3}\left(\frac{2}{d} - 1\right)\right)h_1\\
&\qquad + \frac{1}{f(r)}\partial_t^2 h_1 + \left(\partial_r - \frac{2}{r}\right)\partial_z h_2 - \partial_z^2 h_1\Bigg]X_A,\\
\delta R^{(\text{odd})}_{zA} &= 0 = \frac{1}{2}\Bigg[-\frac{1}{f(r)}\partial_t\partial_z h_0 + \left(f(r)\partial_r + \frac{2}{r}\left(1 - \frac{M}{r}\right)\right)\partial_z h_1\\
&\qquad + \left(\frac{1}{f(r)}\partial_t^2 - \frac{2M}{r^2}\partial_r - f(r)\partial_r^2\right)h_2 + \left(\frac{\ell(\ell+1)}{r^2} + \frac{4M}{dr^3}\right)h_2\Bigg]X_A,\\
\delta R^{(\text{odd})}_{AB} &= 0 = \left[-\frac{1}{f(r)}\partial_t h_0 + \partial_r\left(f(r)h_1\right) + \partial_z h_2\right]X_{AB}.
\end{aligned} \tag{218}$$

[21] The odd-parity perturbations constitute the following symmetric matrix:

$$\gamma_{\mu\nu} = \begin{bmatrix}0 & h_a X_B\\ h_b X_A & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & \gamma_{t\theta} & \gamma_{t\phi}\\ & 0 & 0 & \gamma_{r\theta} & \gamma_{r\phi}\\ & & 0 & \gamma_{z\theta} & \gamma_{z\phi}\\ & \text{sym.} & & 0 & 0\\ & & & & 0\end{bmatrix}. \tag{215}$$

It is evident that we cannot simplify our analysis by treating perturbations that are independent of θ and ϕ, a tactic employed by Gregory [45]; there simply wouldn't be any odd-parity perturbations left, making it rather difficult to prove or disprove isospectrality.

The correct Schwarzschild limit is obtained when we set $d = 2$ and h_2 and z-derivatives to zero. For $d = 3$, we have

$$\begin{aligned}
0 &= f(r)\left(-\partial_r^2 h_0 + \left(\partial_r + \frac{2}{r}\right)\partial_t h_1\right) + \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\left(\frac{2}{d} - 3\right)\right) h_0 + \partial_t\partial_z h_2 - \partial_z^2 h_0, \\
0 &= -\frac{1}{f(r)}\left(\partial_r - \frac{2}{r}\right)\partial_t h_0 + \left(\frac{(\ell-1)(\ell+2)}{r^2} + \frac{2M}{r^3}\left(\frac{2}{d} - 1\right)\right) h_1 + \frac{1}{f(r)}\partial_t^2 h_1 \\
&\quad + \left(\partial_r - \frac{2}{r}\right)\partial_z h_2 - \partial_z^2 h_1, \\
0 &= -\frac{1}{f(r)}\partial_t\partial_z h_0 + \left(f(r)\partial_r + \frac{2}{r}\left(1 - \frac{M}{r}\right)\right)\partial_z h_1 + \left(\frac{1}{f(r)}\partial_t^2 - \frac{2M}{r^2}\partial_r - f(r)\partial_r^2\right) h_2 \\
&\quad + \left(\frac{\ell(\ell+1)}{r^2} + \frac{4M}{dr^3}\right) h_2, \\
0 &= -\frac{1}{f(r)}\partial_t h_0 + \partial_r\left(f(r) h_1\right) + \partial_z h_2.
\end{aligned} \tag{219}$$

One can check that only the second, third and fourth equations are independent. To decouple Eqs.(219), we first rewrite the last equation as

$$\partial_t h_0 = f(r)\partial_r\left(f(r) h_1\right) + f(r)\partial_z h_2 \tag{220}$$

and substitute it into the second and third equations to eliminate any dependence on h_0:

$$\begin{aligned}
0 &= \frac{1}{f(r)}\partial_t^2 h_1 - f(r)\partial_r^2 h_1 + \frac{2(r - 5M)}{r^2}\partial_r h_1 - \partial_z^2 h_1 \\
&\quad + \left(\frac{(\ell-1)(\ell+2)}{r^2} - \frac{56M^2 - 22Mr}{3r^4 f(r)}\right) h_1 - \frac{2M}{r^2 f(r)}\partial_z h_2, \\
0 &= -\frac{2}{r} f(r)\partial_z h_1 - \left[-\frac{1}{f(r)}\partial_t^2 + f(r)\partial_r^2 + \frac{2M}{r^2}\partial_r + \partial_z^2\right] h_2 + \left(\frac{\ell(\ell+1)}{r^2} + \frac{4M}{3r^3}\right) h_2.
\end{aligned} \tag{221}$$

We now introduce the RW function, given by Eq. (90) (it is readily verified from its covariant definition, Eq. (99), that the RW function is identical in four and five dimensions). Moreover, we use that the d'Alembertian operator of $\mathcal{M}^3$ is given by

$$\begin{aligned}
\hat{\Box}\Psi_{\rm RW} &:= \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\, g^{ab}\partial_b \Psi_{\rm RW}\right) \\
&= \left(-\frac{1}{f(r)}\partial_t^2 + f(r)\partial_r^2 + \frac{2M}{r^2}\partial_r + \partial_z^2\right)\Psi_{\rm RW} \\
&= \left(\Box + \partial_z^2\right)\Psi_{\rm RW}.
\end{aligned} \tag{222}$$

We can then write (221) as two coupled equations in terms of $\Psi_{\rm RW}$ and h_2:

$$\hat{\Box}\Psi_{\rm RW} + \frac{2M}{r^3}\partial_z h_2 - \left(\frac{\ell(\ell+1)}{r^2} - \frac{20M}{3r^3}\right)\Psi_{\rm RW} = 0, \tag{223}$$

$$\hat{\Box} h_2 - 2\partial_z \Psi_{\rm RW} - \left(\frac{\ell(\ell+1)}{r^2} + \frac{4M}{3r^3}\right) h_2 = 0. \tag{224}$$

Defining the new potentials V_Ψ and V_2, we can write this more compactly as

$$\left(\hat{\Box} - V_\Psi\right)\Psi_{\rm RW} + \frac{2M}{r^3}\partial_z h_2 = 0, \tag{225}$$

$$\left(\hat{\Box} - V_2\right) h_2 - 2\partial_z \Psi_{\rm RW} = 0, \tag{226}$$

where

$$V_\Psi := \frac{\ell(\ell+1)}{r^2} - \frac{20M}{3r^3}, \tag{227}$$

$$V_2 := \frac{\ell(\ell+1)}{r^2} + \frac{4M}{3r^3}. \tag{228}$$

Interestingly, by applying the same approach as in four-dimensions, we end up with two coupled equations instead of one. In Section 3.7.3, we investigate whether these equations can also be decoupled. First however, we will briefly show that the same results are obtained using a covariant approach.

3.7.2 Decoupling covariantly

We start the covariant approach from the d-dimensional analogue of Eq. (100), i.e.

$$0 = \frac{1}{r}r^a\left(-\Box h_a + \frac{2}{r}r^b\mathscr{D}_a h_b\right) - \frac{2}{r^2}r^a(\mathscr{D}_a\mathscr{D}^b r)h_b - \frac{1}{r^3}\left[2 - \frac{4M}{r}\left(1+\frac{1}{d}\right) - \ell(\ell+1)\right]r^a h_a. \tag{229}$$

The individual terms in this expression can be rewritten using the covariant form of the RW function, Eq. (99). We first term becomes

$$\begin{aligned}\frac{1}{r}r^a\Box h_a &= \Box\left(\frac{1}{r}r^a h_a\right) - \frac{2}{r}(\mathscr{D}^b r^a)\mathscr{D}_b h_a - \frac{1}{r^3}r^a r^b r_b h_a + \frac{1}{r^2}r_b(\mathscr{D}^b r^a)h_a \\ &\quad + \frac{1}{r^2}r^a(\Box r)h_a - \frac{1}{r}(\Box r^a)h_a + \frac{2}{r}r^b\mathscr{D}_b\left(\frac{1}{r}r^a h_a\right) \\ &= \Box\Psi_{\text{RW}} + \frac{2}{r}r^b\mathscr{D}_b\Psi_{\text{RW}} + \frac{2M}{r^3}\partial_z h_2 + \frac{4M}{r^3}\Psi_{\text{RW}}.\end{aligned} \tag{230}$$

The difference compared to the four-dimensional case arises from the fact that this time,

$$-\frac{2}{r}(\mathscr{D}^b r^a)\mathscr{D}_b h_a = -\frac{2M}{r^3}\underbrace{\mathscr{D}^a h_a}_{=0} + \frac{2M}{r^3}\partial_z h_2 = \frac{2M}{r^3}\partial_z h_2. \tag{231}$$

Substituting Eqs. (230) for the first term in Eq. (229) and Eq. (104) for the second term gives

$$\hat{\Box}\Psi_{\text{RW}} + \frac{2M}{r^3}\partial_z h_2 - \left[\frac{\ell(\ell+1)}{r^2} - \frac{4M}{r^3}\left(2 - \frac{1}{d}\right)\right]\Psi_{\text{RW}} = 0, \tag{232}$$

which in the Schwarzschild limit indeed reduces to Eq. (91). For $d = 3$ it reduces to Eq. (223). As in the four-dimensional case, we have now combined three of the vacuum Einstein equations: $\delta R_{tA}^{(\text{odd})} = 0$, $\delta R_{rA}^{(\text{odd})} = 0$ and $\delta R_{AB}^{(\text{odd})} = 0$. A short calculation shows that $\delta R_{zA}^{(\text{odd})} = 0$ indeed gives the additional equation in (224).

3.7.3 Decoupling even further

By explicitly rewriting the system as a set of coupled PDEs, we have shown that it can be reduced to two coupled equations for Ψ_{RW} and h_2, a result we confirmed with a brief covariant calculation. The question remains whether it is possible to decouple them into two independent wave equations.

To this end, we recast Eqs. (223) and (224) into the form

$$\hat{\Box}\vec{v} - \begin{pmatrix} V_\Psi & -\frac{2M}{r^3}\partial_z \\ 2\partial_z & V_2 \end{pmatrix}\vec{v} = 0, \tag{233}$$

where we defined the vector

$$\vec{v} := \begin{pmatrix} \Psi_{\text{RW}} \\ h_2 \end{pmatrix}. \tag{234}$$

In principle we could try to diagonalize the matrix

$$\begin{pmatrix} V_\Psi & -\frac{2M}{r^3}\partial_z \\ 2\partial_z & V_2 \end{pmatrix} \tag{235}$$

by solving an eigenvalue equation. This would lead to a differential equation that requires the determination of boundary conditions in order to find its solution. To avoid this, we decide to adopt a z-dependence of the form e^{ikz} for $\Psi_{\rm RW}$ (or equivalently h_1) and h_2, which is a periodic function in the z-direction with wavenumber[22] k:

$$h_1(t,r,z) = e^{ikz} h_1(t,r), \tag{236}$$
$$h_2(t,r,z) = e^{ikz} h_2(t,r). \tag{237}$$

Such z-dependence is justified by translational symmetry in the z-direction, and is therefore only valid under the assumption that the string has a finite length much shorter than the critical length (see Section 3.2). With this assumption, the system can be written as

$$\Box\vec{v} - A\vec{v} = 0, \tag{238}$$

with

$$A := \begin{pmatrix} k^2 + V_\Psi & -\frac{2ikM}{r^3} \\ 2ik & k^2 + V_2 \end{pmatrix}. \tag{239}$$

Note the change of the dimensionality of the d'Alembertian operator! We can diagonalize the matrix A by decomposing it as

$$A = P^{-1} D P, \tag{240}$$

where the matrix D is diagonal and has the eigenvalues of A as its diagonal elements, and the columns of P consist of the eigenvectors of A. In the script 5D_PERT_COORDINATES.NB, we calculate the eigenvalues and eigenvectors of A, resulting in

$$D = \begin{pmatrix} \lambda_1(r) & 0 \\ 0 & \lambda_2(r) \end{pmatrix}, \tag{241}$$

$$P = \begin{pmatrix} \frac{i(2M+\sqrt{M(4M-k^2r^3)})}{kr^3} & \frac{i(2M-\sqrt{M(4M-k^2r^3)})}{kr^3} \\ 1 & 1 \end{pmatrix}, \tag{242}$$

with eigenvalues

$$\begin{aligned} \lambda_{1,2}(r) &= \frac{1}{2}(V_\Psi + V_2) + k^2 \pm \frac{1}{2}\sqrt{\frac{16k^2M}{r^3} + (V_\Psi^2 - V_2^2)^2} \\ &= \frac{\ell(\ell+1)}{r^2} - \frac{8M}{3r^3} + k^2 \pm \frac{2}{r^3}\sqrt{M(4M+k^2r^3)}. \end{aligned} \tag{243}$$

$\lambda_1(r)$ is attributed to the $-$ sign and $\lambda_2(r)$ to the $+$ sign. Having diagonalized A, we insert Eq. (240) into Eq. (238) and rewrite the result as

$$\begin{aligned} \left(\Box - P^{-1}DP\right)\vec{v} &= 0 \\ (P\Box - DP)\vec{v} &= 0 \\ (P\Box - PD)\vec{v} &= 0 \\ (\Box - D)\vec{v} &= 0. \end{aligned} \tag{244}$$

[22] We are allowed to use the same wavenumber k in both h_1 and h_2 because we are working with linear perturbation theory, where there is no interaction between the modes.

In the first step, we multiplied from the left with P. Then, we commuted the matrices D and P. The latter is allowed because their commutator is, fortunately, zero by direct computation (a highly non-trivial result!). Eq. (244) implies that we can write the system as two independent wave equations,

$$[\Box - \lambda_1(r)]\Psi_{\mathrm{RW}}(t,r) = 0, \tag{245}$$

$$[\Box - \lambda_2(r)]h_2(t,r) = 0, \tag{246}$$

with the functions $\lambda_1(r)$ and $\lambda_2(r)$ as potentials.

The fact that we end up with two independent wave equations in the variables $\Psi_{\mathrm{RW}}(t,r)$ and $h_2(t,r)$ can be attributed to the presence of both radial perturbations and perturbations along the length of the string, with the odd-parity sector of the latter being described by h_2. We will argue in Section 3.9 that the decoupling into two equations in this manner is indeed a plausible result.

3.8 Zerilli equation

In this section, we attempt to find the Zerilli equation in five dimensions. We start from the even-parity vacuum Einstein equations given in Appendix D. For $d = 3$, these are

$$\begin{aligned}
\delta R^{(\mathrm{even})}_{ab} = 0 = \frac{1}{2}\Bigg[&\mathscr{D}_b\mathscr{D}_m f^m_a + \frac{4M}{r^3}\left(f_{ba} - \frac{1}{3}g_{ba}f^m_m\right) + \mathscr{D}_a\mathscr{D}_m f^m_b - \hat{\Box} f_{ab} \\
&+ \frac{2}{r}r_m\left(\mathscr{D}_b f^m_a + \mathscr{D}_a f^m_b - \mathscr{D}^m f_{ab}\right) - \mathscr{D}_a\mathscr{D}_b f^m_m + \frac{\ell(\ell+1)}{r^2}f_{ab} \\
&- \frac{2}{r}(r_b\mathscr{D}_a K - r_a\mathscr{D}_b K) - 2\mathscr{D}_a\mathscr{D}_b K\Bigg]Y, \\
\delta R^{(\mathrm{even})}_{aB} = 0 = \frac{1}{2}\Bigg[&\mathscr{D}_m f^m_a - \mathscr{D}_a f^m_m + \frac{1}{r}r_a f^m_m - \mathscr{D}_a K\Bigg]Y_B, \\
\delta R^{(\mathrm{even})}_{AB} = 0 = \Bigg[&r r_a \mathscr{D}_b f^{ab} - \frac{1}{2}r r^b\mathscr{D}_b f^m_m + r_a r_b f^{ab} + r\mathscr{D}_a r_b f^{ab} - \frac{1}{2}\hat{\Box}(r^2K) \\
&+ \frac{1}{2}\ell(\ell+1)K + \frac{1}{4}\ell(\ell+1)f^a_a\Bigg]\Omega_{AB}Y - \frac{1}{2}f^a_a Y_{AB}.
\end{aligned} \tag{247}$$

3.8.1 Decoupling in coordinates

Eqs. (247) are evaluated using the script 5D_PERT_COORDINATES.NB in the dimension d. This yields a system of ten coupled PDEs with the correct Schwarzschild limit. The equations in terms of d are not particularly illuminating to present. We show the result for $d = 3$:

$$
\begin{aligned}
\delta R^{(\text{even})}_{tt} &= 0 = -\frac{1}{f(r)}\partial_t^2 K + \frac{M}{r^2}\partial_r K - \frac{1}{2f(r)}\partial_t^2 H_0 - \frac{f(r)}{2}\partial_r^2 H_0 - \frac{1}{2}\partial_z^2 H_0 - \frac{r-M}{r^2}\partial_r H_0 + \frac{\ell(\ell+1)}{2r^2}H_0 \\
&\quad + \frac{2r-3M}{r^2 f(r)}\partial_t H_1 + \partial_t\partial_r H_1 - \frac{M}{r^2}\partial_r H_2 - \frac{2M}{r^3}H_2 + \frac{1}{f(r)}\partial_t\partial_z H_3 - \frac{M}{r^2}\partial_z H_4, \\
\delta R^{(\text{even})}_{tr} &= 0 = -\frac{r-3M}{r^2 f(r)}\partial_t K - \partial_t\partial_r K - \frac{1}{2}\partial_t\partial_r H_0 + \frac{M}{2r^2 f(r)}\partial_t H_0 - \frac{1}{2}\partial_z^2 H_1 + \left(\frac{\ell(\ell+1)}{2r^2} - \frac{2M}{r^3}\right)H_1 \\
&\quad + \frac{1}{2}\partial_t\partial_r H_2 + \frac{2r-5M}{2r^2 f(r)}\partial_t H_2 + \frac{1}{2}\partial_r\partial_z H_3 - \frac{M}{r^2 f(r)}\partial_z H_3 + \frac{1}{2}\partial_t\partial_z H_4, \\
\delta R^{(\text{even})}_{tz} &= 0 = -\partial_t\partial_z K - \frac{1}{2}\partial_t\partial_z H_0 + \frac{f(r)}{2}\partial_r\partial_z H_1 + \frac{f(r)}{r}\partial_z H_1 - \frac{f(r)}{2}\partial_r^2 H_3 - \frac{f(r)}{r}\partial_r H_3 \\
&\quad + \left(\frac{\ell(\ell+1)}{2r^2} + \frac{M}{r^3}\right)H_3 + \frac{f(r)}{r}\partial_t H_4 + \frac{f(r)}{2}\partial_t\partial_r H_4 + \frac{1}{2}\partial_t\partial_z H_5, \\
\delta R^{(\text{even})}_{rr} &= 0 = -f(r)\partial_r^2 K - \frac{M}{r^2}\partial_r K - \frac{M}{r^2 f(r)}\partial_t H_0 + \frac{M}{r^2}\partial_r H_0 - \frac{2M}{r^3}H_0 - \partial_t\partial_r H_1 + \frac{1}{2f(r)}\partial_t^2 H_2 \\
&\quad + \frac{f(r)}{2}\partial_r^2 H_2 + \frac{r-M}{r^2}\partial_r H_2 - \frac{1}{2}\partial_z^2 H_2 + \frac{\ell(\ell+1)}{2r^2}H_2 + f(r)\partial_r\partial_z H_4 + \frac{M}{r^2}\partial_z H_4, \\
\delta R^{(\text{even})}_{rz} &= 0 = -\partial_r\partial_z K + \frac{1}{r}\partial_z K + \frac{M}{2r^2 f(r)}\partial_z H_0 - \frac{1}{2f(r)}\partial_t\partial_z H_1 + \frac{1}{2}\partial_r\partial_z H_2 + \frac{2r-3M}{2r^2 f(r)}\partial_z H_2 \\
&\quad - \frac{1}{2f(r)}\partial_t\partial_r H_3 + \frac{1}{2r^2 f(r)}\partial_t^2 H_4 + \left(\frac{\ell(\ell+1)}{2r^2} + \frac{M}{r^3}\right)H_4 + \frac{1}{2}\partial_r\partial_z H_5, \\
\delta R^{(\text{even})}_{zz} &= 0 = \partial_z^2 K + \frac{1}{f(r)}\partial_t\partial_z H_3 - f(r)\partial_r\partial_z H_4 - \frac{2(r-M)}{r^2}\partial_z H_4 - \frac{1}{2f(r)}\partial_t^2 H_5 + \frac{f(r)}{2}\partial_r^2 H_5 \\
&\quad + \frac{r-M}{r^2}\partial_r H_5 - \frac{1}{2}\partial_z^2 H_5 - \left(\frac{\ell(\ell+1)}{2r^2} + \frac{2M}{r^3}\right)H_5, \\
\delta R^{(\text{even})}_{tA} &= 0 = \partial_t K + \partial_t H_0 - f(r)\partial_r H_1 - \frac{2M}{r^2}H_1 - \partial_z H_3, \\
\delta R^{(\text{even})}_{rA} &= 0 = \partial_r K - \frac{M}{r^2 f(r)}H_0 + \frac{1}{f(r)}\partial_t H_1 - \partial_r H_2 - \frac{M}{r^2 f(r)}H_2 - \partial_z H_4, \\
\delta R^{(\text{even})}_{zA} &= 0 = \partial_z K + \frac{1}{f(r)}\partial_t H_3 - f(r)\partial_r H_4 - \frac{2M}{r^2}H_4 - \partial_z H_5.
\end{aligned} \tag{248}
$$

$\delta R^{(\text{even})}_{AB} = 0$ yields two conditions, namely

$$
\begin{aligned}
0 &= \frac{1}{f(r)}\partial_t^2 K - f(r)\partial_r^2 K - \partial_z^2 K + \frac{(6M-4r)}{r^2}\partial_r K + \frac{(\ell+2)(\ell-1)}{r^2}K \\
&\quad - \frac{2}{r}\partial_t H_1 + \frac{2}{r}f(r)\partial_r H_2 + \frac{2}{r^2}H_2 + \frac{2}{r}f(r)\partial_z H_4
\end{aligned} \tag{249}
$$

and the trace condition

$$
f^a_a = -H_0 + H_2 + H_5 = 0 \longrightarrow H_5 = H_0 - H_2. \tag{250}
$$

With this trace condition we can eliminate H_5 from the resulting system.

We observe that the system consists of three first-order equations and seven second-order equations. In principle, we should be able to derive an algebraic relation from the second-order equations following the method from Section 2.8. However, this turns out to be practically unfeasible. Eliminating the various (mixed) derivatives with the z-coordinate is extremely difficult and boils down to brute-force trial and error[23].

A possible method to simplify the problem is to assume an exponential dependence for the modes, for example, in the variables t and z, effectively disposing of the corresponding partial derivatives. While this simplifies the problem to some degree, it remains insufficient to obtain a single algebraic identity. Even if such an identity were found in this case, this is just the starting point of the decoupling process in coordinates! Clearly, a more systematic approach is required to decouple the even-parity perturbation equations.

3.8.2 Decoupling covariantly

A more systematic approach to the decoupling problem is provided by Martel [9], which uses covariant calculations. While this approach is more organized in principle, it does come with its own set of challenges. As we will see, the method is highly detailed and specifically designed to work in four dimensions. We must be very careful in identifying the additional terms that will appear in our five-dimensional calculations. Furthermore, although the ZM function in (142) was successful in decoupling the four-dimensional system, there is no guarantee that the same function will work in five dimensions.

As in Section 3.8.2, we make use of the Einstein tensors in the linearized vacuum Einstein equations. They are given in terms of even-parity spherical harmonics in Appendix D, Eqs. (317)-(319). A minor difference with the four-dimensional case is that we cannot make use of Eq. (30) to simplify Q_{ab}. The five-dimensional analogue of Eqs. (151)-(153) is therefore

$$\begin{aligned}
0 = Q_{ab} &= \mathscr{D}_c\mathscr{D}_{(b}f^c_{a)} - \frac{1}{2}g_{ab}\mathscr{D}_c\mathscr{D}_d f^{cd} - \frac{1}{2}\hat{\Box} f_{ab} + \frac{2}{r}r^c\left(\mathscr{D}_{(b}f_{a)c} - g_{ab}\mathscr{D}_d f^d_c\right) - \frac{r^c}{r}\mathscr{D}_c f_{ab} \\
&\quad + \frac{\lambda+1}{r^2}f_{ab} - \frac{1}{r^2}g_{ab}r^c r^d f_{cd} - \frac{1}{r}g_{ab}(\mathscr{D}_c r_d)f^{cd} - \mathscr{D}_a\mathscr{D}_b K + g_{ab}\hat{\Box}K \\
&\quad - \frac{2}{r}r_{(a}\mathscr{D}_{b)}K + \frac{3}{r}g_{ab}r^c\mathscr{D}_c K - \frac{\lambda}{r^2}g_{ab}K, && (251) \\
0 = Q_a &= \mathscr{D}_b f^b_a - \mathscr{D}_a K, && (252) \\
0 = Q^\flat &= -\mathscr{D}_a\mathscr{D}_b f^{ab} - \frac{2}{r}r_a\mathscr{D}_b f^{ab} + \Box K + \frac{2}{r}r^a\mathscr{D}_a K. && (253)
\end{aligned}$$

From this point onward, we follow the steps outlined in Section 2.8.2, with the necessary adaptation to Q_{ab}. The first objective is again to rewrite Eqs. (252)-(253) into three differential equations in terms of only K and v and to decouple them using the ZM function. We keep the calculations in terms of the dimension d and only insert $d = 3$ at the end.

The first equation in the new system is obtained by taking the trace of Q_{ab}, and is the analogue of Eq. (154). Making use of the trace condition, this yields

$$\begin{aligned}
0 = Q^a_a &= \hat{\Box}K + \frac{2}{r}r^a\mathscr{D}_a K - \frac{2\lambda}{r^2}K - \frac{2}{r^2}r^a r^b f_{ab} - \frac{2}{r}(\mathscr{D}_c r_d)f^{cd} \\
&= \hat{\Box}K - \frac{2\lambda}{r^2}K - \frac{2}{r^2}v + \frac{2M}{r^3}f(r)H_5. && (254)
\end{aligned}$$

The second and third equations in our system are constructed by taking combinations of Q_{ab}. Unfortunately, at this point we cannot make use of relation (157) due to the fact that Eq. (156) cannot simply be extended from two to three dimensions. Due to the additional degrees of freedom in the Riemann tensor, the three-dimensional analogue of Eq. (157) must include extra terms, though their exact form remains unclear[24]. This presents an inconvenience for which we currently have no solution.

[23] In the script 5D_PERT_COORDINATES.NB we developed a method to manipulate the equations, allowing the brave reader to give it a try.

Continuing without this simplification, the first combination is expressed as

$$\begin{aligned} 0 = f(r)Q^a{}_a - r^a r^b Q_{ab} = &-\frac{r^a r^b}{r}(r^c \mathscr{D}_c f_{ab} - r\mathscr{D}_a\mathscr{D}_b K) - \left[\frac{\lambda+1}{r^2} + \frac{f(r)}{r^2}\right] r^a r^b f_{ab} \\ &+ \frac{f(r)}{r} r^a \mathscr{D}_a K - \frac{\lambda}{r^2} f(r) K - \frac{f(r)}{r}(\mathscr{D}_c r_d) f^{cd} - r^a r^b \mathscr{D}_c \mathscr{D}_a f^c_b \\ &+ \frac{f(r)}{2}\mathscr{D}_c\mathscr{D}_d f^{cd} + \frac{1}{2} r^a r^b \hat{\Box} f_{ab} + \frac{2}{r} f(r) r_c \mathscr{D}_c \mathscr{D}_d f^{cd}. \end{aligned} \tag{255}$$

We make use of the equivalent of Eq. (164), i.e.

$$\begin{aligned} r^a r^b \hat{\Box} f_{ab} = &\hat{\Box} v + \frac{4M}{r^3} v + r\hat{\Box}(r^a \mathscr{D}_a K) + 2 r^a r^b \mathscr{D}_a \mathscr{D}_b K \\ &+ \frac{4M}{r^2} r^a \mathscr{D}_a K + \frac{4M}{r^2} f(r)\partial_z H_4 + \frac{2M^2}{r^4} H_5, \end{aligned} \tag{256}$$

and rewrite Eq. (255) in the same manner as in Section 2.8.2:

$$\begin{aligned} 0 = &-\frac{1}{r} r^a \mathscr{D}_a v - \frac{1}{r^2}\left[\lambda + 2 + \frac{2M}{dr}(d-4)\right] v - \frac{1}{r}\left(\lambda + \frac{7M}{r} - \frac{8M}{dr}\right) r^a \mathscr{D}_a K \\ &- \frac{\lambda}{r^2} f(r) K + \frac{f(r)}{2}\hat{\Box} K + \frac{1}{2}\hat{\Box} v + \frac{1}{2}\hat{\Box}(r^a \mathscr{D}_a K) + \frac{2M}{r^2} f(r) \partial_z H_4 + \frac{M}{r^3}\left(1 - \frac{M}{r}\right) H_5. \end{aligned} \tag{257}$$

This is the second equation of our new system.

We finally make the following combination:

$$\begin{aligned} 0 = \frac{2}{r} r^a r^b Q_{ab} + r^b \mathscr{D}_b Q^a_a = &-\frac{1}{r} r^a r^b \hat{\Box} f_{ab} - \frac{2}{r} r^c (\mathscr{D}_c \mathscr{D}_a r_b) f^{ab} - \frac{2}{r} r^c (\mathscr{D}_a r_b) \mathscr{D}_c f^{ab} \\ &+ \left(\frac{2(\lambda+2)}{r^3} - \frac{8M(d-2)}{dr^4}\right) r^a r^b f_{ab} + \frac{f(r)}{r}\hat{\Box} K + r^a \mathscr{D}_a\left(\hat{\Box} K\right) \\ &+ \frac{2}{r^2}\left(\frac{5M}{r} - 2 - \lambda\right) r^a \mathscr{D}_a K + \frac{2}{r} r^a r^b \mathscr{D}_a \mathscr{D}_b K + \frac{2\lambda}{r^3} f(r) K. \end{aligned} \tag{258}$$

To simplify this expression, we make use of the following observations:

- We make use of Eq. (256) to rewrite the first term.
- We cannot make use of Eq. (30), and must evaluate in Mathematica the second and third term[25] as

$$-\frac{2}{r} r^c (\mathscr{D}_c \mathscr{D}_a r_d) f^{ad} = -\frac{4M}{r^4} f(r)(H_0 - H_2) = -\frac{4M}{r^4} f(r) H_5, \tag{259}$$

$$-\frac{2}{r} r^c (\mathscr{D}_a r_d) \mathscr{D}_c f^{ad} = \frac{2M}{r^3} f(r) \partial_r H_5. \tag{260}$$

[24] One might argue that Eq. (21) gives a straightforward extension for $d = 3$, namely

$$\mathscr{R}_{ab} = \frac{\mathscr{R}}{3} g_{ab}.$$

However, this is only valid for the unperturbed background spacetime. In Section 2.8.2 we were allowed to use Eq. (156) because it holds *in general* for a two-dimensional manifold.

[25] Indeed, by taking this approach, we have strayed from the covariant method. However, while it is possible to retain these terms in their original form, expressing them in terms of H_5 allows us to explicitly recognize them as "new" contributions, which we prefer.

- Moreover, we rewrite

$$
\begin{aligned}
r^a \mathscr{D}_a \left(\hat{\Box} K\right) &= g^{bc} r^a \left(\mathscr{D}_c \mathscr{D}_a \mathscr{D}_b K + R_{acb}{}^d \mathscr{D}_d K\right) \\
&= g^{bc} \mathscr{D}_c (r^a \mathscr{D}_b \mathscr{D}_a K) - g^{bc} (\mathscr{D}_c r^a) \mathscr{D}_a \mathscr{D}_b K + \frac{4M}{dr^3} r^a (g_{ab} g^{bd} - g_a^d g_b^b) \mathscr{D}_d K \\
&= g^{bc} \mathscr{D}_c \mathscr{D}_b (r^a \mathscr{D}_a K) - g^{bc} \mathscr{D}_c \left((\mathscr{D}_b r^a) \mathscr{D}_a K\right) - g^{bc} (\mathscr{D}_c r^a) \mathscr{D}_a \mathscr{D}_b K + \frac{4M}{dr^3} (1-d) r^a \mathscr{D}_a K \\
&= \hat{\Box}(r^a \mathscr{D}_a K) + \frac{2M}{r^3} r^a \mathscr{D}_a K - 2(\mathscr{D}^b r^a) \mathscr{D}_a \mathscr{D}_b K + \frac{4M}{dr^3} (1-d) r^a \mathscr{D}_a K \\
&= \hat{\Box}(r^a \mathscr{D}_a K) - \frac{2M}{r^2} \hat{\Box} K + \frac{4M}{dr^3} \left(1 - \frac{d}{2}\right) r^a \mathscr{D}_a K + \frac{2M}{r^2} \partial_z^2 K. \qquad (261)
\end{aligned}
$$

In the third line we again used Mathematica to evaluate that

$$
g^{bc} (\mathscr{D}_c \mathscr{D}_b r^a) = -\frac{2M}{r^3} r^a, \qquad (262)
$$

and we used that

$$
(\mathscr{D}^b r^a) \mathscr{D}_a \mathscr{D}_b K = \frac{M}{r^2} \hat{\Box} K - \frac{M}{r^2} \partial_z^2 K \qquad (263)
$$

in the fifth line. In the Schwarzschild limit, the third and fourth term in Eq. (261) are zero, yielding exactly Eq. (166).

Applying these results to Eq. (258) finally yields

$$
\begin{aligned}
0 = &-\frac{1}{r} \hat{\Box} v + \frac{2}{r^3} \left[\lambda + 2 - \frac{2M}{r} \left(3 - \frac{4}{d}\right)\right] v + \frac{1}{r} \left(1 - \frac{4M}{r}\right) \hat{\Box} K \\
&+ \frac{4M}{r^3} \left(\frac{5}{d} - 1\right) r^a \mathscr{D}_a K + \frac{2\lambda}{r^3} f(r) K + \frac{2M}{r^2} \partial_z^2 K - \frac{4M}{r^3} f(r) \partial_z H_4 \\
&+ \frac{2M}{r^3} f(r) \partial_r H_5 - \frac{2M}{r^4} \left(2 - \frac{3M}{r}\right) H_5. \qquad (264)
\end{aligned}
$$

In the Schwarzschild limit, this correctly reduces to Eq. (167).

Combining our results, we see that we have obtained a system in terms of the variables v, K, H_4 and H_5:

$$
0 = \hat{\Box} K - \frac{2\lambda}{r^2} K - \frac{2}{r^2} v + \frac{2M}{r^3} f(r) H_5, \qquad (265)
$$

$$
\begin{aligned}
0 = &-\frac{1}{r} r^a \mathscr{D}_a v - \frac{1}{r^2} \left[\lambda + 2 + \frac{2M}{dr} (d-4)\right] v - \frac{1}{r} \left(\lambda + \frac{7M}{r} - \frac{8M}{dr}\right) r^a \mathscr{D}_a K \\
&- \frac{\lambda}{r^2} f(r) K + \frac{f(r)}{2} \hat{\Box} K + \frac{1}{2} \hat{\Box} v + \frac{1}{2} \hat{\Box}(r^a \mathscr{D}_a K) + \frac{2M}{r^2} f(r) \partial_z H_4 + \frac{M}{r^3} \left(1 - \frac{M}{r}\right) H_5, \qquad (266)
\end{aligned}
$$

$$
\begin{aligned}
0 = &-\frac{1}{r} \hat{\Box} v + \frac{2}{r^3} \left[\lambda + 2 - \frac{2M}{r} \left(3 - \frac{4}{d}\right)\right] v + \frac{1}{r} \left(1 - \frac{4M}{r}\right) \hat{\Box} K + \frac{4M}{r^3} \left(\frac{5}{d} - 1\right) r^a \mathscr{D}_a K \\
&+ \frac{2\lambda}{r^3} f(r) K + \frac{2M}{r^2} \partial_z^2 K - \frac{4M}{r^3} f(r) \partial_z H_4 + \frac{2M}{r^3} f(r) \partial_r H_5 - \frac{2M}{r^4} \left(2 - \frac{3M}{r}\right) H_5. \qquad (267)
\end{aligned}
$$

Eliminating v in favour of $\Psi_{\rm ZM}$ gives us the five-dimensional equivalent of system (169):

$$0 = \hat{\Box} K + \frac{6M}{r^3}K - \frac{2\Lambda}{r^3}(\lambda+1)\Psi_{\rm ZM} + \frac{2M}{r^3}f(r)H_5, \tag{268}$$

$$\begin{aligned}0 = &\left[-\frac{\Lambda(\lambda+1)}{r^2} + \frac{30M^2 + 6Mr(\lambda-2) - 2\lambda r^2}{r^4 f(r)}\right] r^a \mathscr{D}_a \Psi_{\rm ZM} \\ &+ \frac{1}{r^2}\left[\lambda(\lambda+1) + \frac{\lambda M}{r}\left(7 - \frac{8}{d}\right) + \frac{6M}{r} + \frac{6M^2}{r^2}\left(1 - \frac{4}{d}\right)\right]\Psi_{\rm ZM} - \frac{\Lambda\lambda(\lambda+1)}{2r}\hat{\Box}\Psi_{\rm ZM} \\ &+ \frac{f(r)-\Lambda}{2}\hat{\Box}K + \frac{1}{2}\hat{\Box}(r^a\mathscr{D}_a K) - \frac{1}{r}\left[\lambda - \frac{M}{r}\left(3 + \frac{8}{d}\right) + \frac{7M}{r^3}\right] r^a \mathscr{D}_a K \\ &+ \frac{1}{r^2}\left[\lambda(\lambda+1) + \frac{\lambda M}{r}\left(7 - \frac{8}{d}\right) + \frac{3M}{r} + \frac{3M^2}{r^2}\left(5 - \frac{8}{d}\right)\right] K \\ &+ \frac{2M}{r^2}f(r)\partial_z H_4 + \frac{M}{r}\left(1 - \frac{M}{r}\right)H_5,\end{aligned} \tag{269}$$

$$\begin{aligned}0 = &-\frac{\Lambda}{r^2}(\lambda+1)\hat{\Box}\Psi_{\rm ZM} + \frac{2}{r^3}\left(\lambda + \frac{6M}{r}\right)(\lambda+1) r^a\mathscr{D}_a\Psi_{\rm ZM} \\ &+ \frac{2}{r^4}\left[\lambda(\lambda+1) + \frac{M}{dr}(8\lambda - 3d) + \frac{6M^2}{dr^2}(4+d)\right](\lambda+1)\Psi_{\rm ZM} + \frac{1}{r}\left(\lambda + 1 - \frac{M}{r}\right)\hat{\Box}K \\ &+ \frac{10M}{r^3}(d-2)r^a\mathscr{D}_a K - \frac{2}{r^3}\left[\lambda(\lambda+1) + \frac{M}{r}\left(3 + \frac{8\lambda}{d} - \lambda\right) + \frac{3M^2}{r^2}\left(\frac{8}{d} - 3\right)\right] K \\ &+ \frac{2M}{r^2}\partial_z^2 K - \frac{4M}{r^3}f(r)\partial_z H_4 + \frac{2M}{r^3}f(r)\partial_r H_5 - \frac{2M}{r^4}\left(2 - \frac{3M}{r}\right)H_5.\end{aligned} \tag{270}$$

The fact that it is not possible to use simplification (30) proves to be a significant obstacle. In four dimensions, this simplification allowed us to eliminate K from the system, leaving a single equation in terms of $\Psi_{\rm ZM}$, namely the Zerilli equation. However, in five dimensions, it turns out that K cannot be completely removed from Eqs. (268)-(270) due to certain terms in the second equation that resist simplification or convenient reformulation. As a result, the covariant approach used in Section 2.8.2 does not produce the desired outcome in five dimensions.

Eqs. (268)-(270) do not encapsulate all the information originally present in the even-parity perturbation equations. For completeness, we conclude this section by compiling the equations that contain indispensable information about the perturbation equations.

First, we observe that Eqs. (268)-(270) are formed from the combinations

$$\begin{aligned}Q^a_a &= g^{ab}Q_{ab} = -\frac{1}{f(r)}Q_{tt} + f(r)Q_{rr} + Q_{zz}, \\ r^a r^b Q_{ab} &= r^r r^r Q_{rr} = Q_{rr},\end{aligned}$$

therefore incorporating only information from the trace part of Q_{ab}, i.e. Q_{tt}, Q_{rr} and Q_{zz}. To account for the missing information, we must also include the components $Q_{tr} = 0$, $Q_{tz} = 0$, $Q_{rz} = 0$ and $Q_{zB} := Q_z = 0$. To this end, we evaluate the tr, tz and rz components of Eq. (251) and the z component of Eq. (252) in the script 5D_PERT_COORDINATES.NB. We find that it is relatively straightforward to combine $Q_{tr} = 0$, $Q_{tz} = 0$ and $Q_{rz} = 0$ into one second-order, semi-algebraic equation (with only r-derivatives) in terms of the variables H_0, H_1, H_2, H_3 and H_4 if we assume an exponential t- and z-dependence. The explicit calculation is worked out in the script and not particularly illuminating to present. We state the final result:

$$
\begin{aligned}
0 = {} & f(r)\partial_r H_0 + \frac{M}{r^2}H_0 + \frac{iM}{\omega r^2}f(r)\partial_r H_1 + \frac{i}{\omega}\left[\frac{k^2-\omega^2}{2} - \frac{Mk^2}{r} + \frac{\lambda+1}{r^2} - \frac{2M}{r^3}\left(\lambda + \frac{M}{r}\right)\right]H_1 \\
& - \frac{f(r)}{2}\partial_r H_2 + \frac{M}{r^2}H_2 - \frac{M}{k\omega r^2}f(r)\partial_r^2 H_3 - \frac{1}{\omega}\left[\frac{k^2+\omega^2}{2k} - \frac{Mk}{r} + \frac{2M}{kr^3}f(r)\right]\partial_r H_3 \\
& + \frac{M}{k\omega r^4}\left[2(\lambda+1) + kr^2 - \frac{\omega^2 r^2}{f(r)}\right]H_3 \\
& - \frac{i}{k}\left[\frac{k^2-\omega^2}{2} - \frac{Mk^2}{r} + \frac{\lambda+1}{r^2} - \frac{2M}{r^3}\left(\lambda + 2 - \frac{3M}{r}\right)\right]H_4
\end{aligned} \tag{271}
$$

$$
Q_z = \partial_z K - \frac{1}{f(r)}\partial_t H_3 + f(r)\partial_r H_4 + \frac{2M}{r^2}H_4 + \partial_z H_5 \tag{272}
$$

Finally, we observe that the ZM function encodes information from both K and H_2 when we express v in coordinates:

$$
\Psi_{\mathrm{ZM}} = \frac{r}{\lambda+1}\left[K + \frac{v}{\Lambda}\right] = \frac{r}{\lambda+1}\left[K + f(r)\frac{H_2 - r\partial_r K}{\Lambda}\right]. \tag{273}
$$

This means that in total our new system consists of the six variables $K, H_0, H_1, H_2, H_3, H_4$ in six equations, Eqs. (268)-(273). Therefore, all information from the original system, Eqs. (145)-(148), is retained.

3.9 Overview of variables in four and five dimensions

We have seen that keeping track of the number of (independent) variables and equations in the odd- and especially the even-parity systems of perturbation equations can be challenging. To conclude this section, we provide a comparison of the number of variables, degrees of freedom (DOF), and equations in four and five dimensions.

The initial number of variables in four and five dimensions, that is, the total number of modes present in the spherical harmonics decomposition, is presented in Table 2.

	Schwarzschild	**Black string**
Odd-parity	h_0, h_1, h_2 (3)	h_0, h_1, h_2, h_3 (4)
Even-parity	$H_0, H_1, H_2, j_0, j_1, K, G$ (7)	$H_0, H_1, H_2, H_3, H_4, H_5, j_0, j_1, j_2, K, G$ (11)
Total variables	10	15

Table 2: Initial (number of) variables for both the Schwarzschild (4d) and black string (5d) spacetimes, for both parities.

We can verify the total number of variables by considering that the perturbing metric, $\gamma_{\mu\nu}$, is a symmetric rank-2 tensor. In a d-dimensional spacetime, such a tensor has

$$
\frac{d(d+1)}{2} \tag{274}
$$

independent components.

We have seen that the gauge vector Ξ_μ removes d variables, since it has d independent components. Additionally, the Bianchi identities (which we did not discuss in this thesis) impose d constraints on the vacuum Einstein equations, thus removing another d variables. Therefore, we are left with

$$
\frac{d(d+1)}{2} - 2d \tag{275}
$$

independent variables, also called the physical (or *dynamical*) DOF. This is also the minimal number of equations required to describe the perturbations. In four dimensions, the number of dynamical DOF is therefore two; Ψ_{RW} and Ψ_{ZM}. Both are tensor modes (on $\mathcal{M}^4$, which should not be confused with the spherical harmonic tensor modes on $\mathcal{S}^2$) and fully describe gravitational waves emerging from perturbations of the spacetime. We confirmed this in the first part of this thesis, when we derived that the perturbations are fully described by two decoupled wave equations (the RW and Zerilli equations) in terms of these functions. Based on Eq. (275), in five dimensions we expect to have five dynamical DOF, which in this case will likely be two tensor modes, two vector modes and one scalar mode[26]. The (expected) DOF in four and five dimensions are summarized in Table 3.

	Schwarzschild	**Black string**
Odd-parity	Ψ_{RW} (1)	Ψ_{RW}, h_2 (2)
Even-parity	Ψ_{ZM} (1)	Ψ_{ZM}, vector mode, scalar mode (3)
Dynamical DOF	2	5

Table 3: Dynamical DOF for both the Schwarzschild (4d) and black string (5d) spacetimes, for both parities.

In Section 3.7.3 we have seen that indeed the five-dimensional odd-parity perturbations are described by the independent modes Ψ_{RW} and h_2. While we have not been able to reduce the even-parity sector to three variables, we have reason to expect that it decouples into three independent equations, one in terms of a tensor mode, one in a vector mode and one in a scalar mode. *If* isospectrality were to hold in the black string spacetime, it would be most logical that it is proven by relating the odd tensor mode to the even tensor mode and the odd vector mode to the even vector mode, both via a proper Darboux transform, while the scalar mode fully decouples. The tensor mode is likely to be the ZM function, but it is not strictly necessary that it has exactly the form of Eq. (273). The specific vector and scalar modes involved in these decoupled equations cannot be determined with the information we currently have. The precise identification of these modes would require a more detailed analysis of the decoupling of the perturbation equations in the even-parity sector.

[26]Credits to my supervisor B. Bonga for pointing this out in one of our discussions.

4 Conclusion

We have provided a comprehensive and self-consistent review of metric perturbation theory of the Schwarzschild spacetime. Scattered information from the literature was combined, and we verified the well-known result that the RW and Zerilli equations are related by a Darboux (or Chandrasekhar) transformation. We have also shown that this transformation implies that the transmission and reflection coefficients of the Darboux-related RW and Zerilli potentials are equal, and hence that the QNM spectra of both potentials coincide. The purpose of this was to establish the groundwork for exploring whether isospectrality would hold in a five-dimensional context.

In the second part of this thesis, we extended the metric perturbation theory framework to the black string spacetime, a five-dimensional counterpart of the Schwarzschild black hole. This was achieved by introducing an extra spatial dimension, independent of the coordinates. By incorporating this additional coordinate into x^a (thus transforming $\mathcal{M}^2$ to $\mathcal{M}^3$), we anticipated that the formalism would seamlessly adapt to perturbations of this five-dimensional spacetime, introducing only a modest increase in complexity. Indeed, we found that the decomposition into spherical harmonics could be straightforwardly extended by adding new variables into the even-parity modes f_{ab} and j_a, and the odd-parity mode h_a. In the RW gauge, we were able to eliminate four even-parity mode components (j_a and G) and one odd-parity mode component (h_3), analogous to the four-dimensional case. From there, we calculated the odd- and even-parity vacuum Einstein equations from the linearized Ricci (and Einstein) tensors, which we could treat separately. These systems of equations could then be decoupled in two ways: either by working in coordinates and expressing the system as a set of coupled second-order PDEs, or by employing a covariant approach with the RW and ZM functions in their covariant form.

For the odd-parity case, we applied both methods and, due to the manageable number of perturbation equations and modes, were able to follow the same steps as in four dimensions without encountering significant difficulty. Interestingly, instead of deriving a single RW equation solely in terms of the RW function, we ended up with two coupled equations involving both the RW function and h_2. The covariant approach led to the same conclusion. The structure of the equations allowed for decoupling, but this process required a few non-trivial steps. By rewriting the two equations in vector-matrix form and assuming an exponential t- and z-dependence for the RW function and h_2, we were able to construct a diagonal matrix D with relatively simple eigenvalues on the diagonal. We discovered that the matrix $A = P^{-1}DP$ decouples the system, allowing us to write down two fully independent wave equations for the RW function and h_2. This hinged on the fact that D and P commute, which seemed like a mere coincidence, as there is no obvious a priori condition (that we know of) that would guarantee such commutation.

In the even-parity case, the introduction of new variables and perturbation equations presented substantial challenges. Decoupling the system of vacuum Einstein equations in coordinates proved to be exceedingly difficult due to the large number of equations (10), variables (7), and the numerous mixed derivatives involving the z-coordinate. Although we were able to reduce the number of variables from seven to six by eliminating H_5 in favour of H_0 and H_2 (thanks to the new trace condition) this simplification did not significantly ease the process. An attempt was made in the MATHEMATICA script 5D_PERT_COORDINATES.NB to make the equations semi-algebraic by introducing an exponential t- and z-dependence, but fully decoupling the equations remained a futile attempt. An effort to decouple the system covariantly by following the detailed four-dimensional calculations in Martel [9] also did not yield the desired result.

It is clear that we have not been able to definitively prove or disprove isospectrality of the QNM frequencies of black strings. While the formalism we used is well-suited for extending the submanifold $\mathcal{M}^2$ to higher dimensions, the calculations have shown to become considerably more complex when adding just one extra dimension without any dependence on the existing coordinates, which is particularly evident in the even-parity case. However, we do not rule out the possibility that isospectrality could hold in the black string spacetime. We anticipate that a proper decoupling of the even-parity system will lead to three independent equations: one for $\Psi_{\rm ZM}$, one for a vector mode, and one for a scalar mode. It may be possible to relate the odd tensor mode to the even tensor mode, and the odd vector mode to the even vector mode, with the the scalar mode decoupling entirely.

Throughout this thesis, we have refrained from making any assumptions regarding the type of perturbations we considered. In hindsight, one could propose that it is possible to focus on less general perturbations. For instance, Gregory [1] considers only spherically symmetric perturbations, which involve no cross terms with the angular coordinate in $\gamma_{\mu\nu}$. Such a simplification makes it impossible to prove isospectrality, as the odd-parity sector of the spherical harmonics would trivially vanish (see the footnote on page 39).

A potentially simpler approach could be to consider radial perturbations, i.e., perpendicular to the string's axis and therefore independent of the z-coordinate. While this might simplify the perturbation equations, it makes us blind to potentially interesting effects that could arise from perturbations along the z-direction. Even if one were able to prove isospectrality in this restricted case, it is quesionable whether this is actually a useful result. Real perturbations of black strings will almost never be fully independent of the z-direction. Therefore, demonstrating isospectrality for a very limited subset of perturbations would not serve as conclusive evidence for isospectrality as a *general* property of black strings. In this light, we believe it is crucial to avoid making simplifying assumptions about the types of perturbations considered.

We should also acknowledge that we did not make use of the Bianchi identities in our calculations. These identities impose additional constraints on the perturbation equations and could potentially assist in decoupling the even-parity system of vacuum Einstein equations by reducing the number of independent variables. For future investigations, we recommend considering the application of the three even-parity Bianchi identities, as described by Martel's [9]. They may be a valuable tool for simplifying and further advancing the analysis of the even-parity sector.

It is also possible that a different function, other than the covariant ZM function given in Eq. (142), decouples the five-dimensional system. However, the literature on systematic methods for identifying such a decoupling function is quite limited, and finding one may require a challenging process of trial and error. (see [14] for finding the Zerilli function in a slightly different context).

In some last remarks, we would like to acknowledge that the formalism of metric perturbations is limited by factors that we have not touched upon in detail. While these limitations do not directly affect the proof of isospectrality for a given spacetime, they are important to keep in mind. For example, as noted in Section 2.5, in the spherical harmonic decomposition restricts the sum over the mode label ℓ to $\ell \geq 2$, since $\ell = 0$ and $\ell = 1$ modes are non-radiative and require special treatment. Additionally, we have worked within the framework of *linear* perturbation theory, which may overlook physical phenomena present at higher order.

Decoupling the even-parity perturbation equations of the black string may hold the key to proving isospectrality in this five-dimensional spacetime. For now however, the answer remains elusive.

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A Appendix: Notation conversion table

We chose to adopt a slightly different notation than Martel [9] and Martel & Poisson [21], of which we make use extensively in this thesis. In order to translate between this thesis and their works, we list the most important notational differences in the table below.

Quantity	This thesis	Martel [9]	Martel & Poisson [21]
Partial derivative	∂	,	∂
Covariant derivative of $g_{\mu\nu}$	∇	;	Not defined
Covariant derivative of g_{ab}	$\mathscr{D}$	:	∇
Covariant derivative of Ω_{AB}	D	$\vert$	D
Perturbation metric	$\gamma_{\mu\nu}$	$h_{\mu\nu}$	$p_{\mu\nu}$
Even-parity field	f_{ab}	p_{ab}	h_{ab}
Even-parity field	j_a	q_a	j_a
Vector harmonic	Y_A	Z_A	Y_A
Vector harmonic	$\Omega_{AB}Y$	U_{AB}	$\Omega_{AB}Y$
Tensor harmonic	Y_{AB}	V_{AB}	Y_{AB}
Tensor harmonic	X_{AB}	W_{AB}	X_{AB}

Table 4: Notational differences for (covariant) derivatives, perturbation quantities and spherical harmonics. Martel's derivatives are written as subscripts. We chose to adopt the same notation as Martel & Poisson for the spherical harmonics.

B Appendix: Spherical Harmonics

Spherical harmonics are a special type of functions defined on the surface of the two-sphere ($\mathcal{S}^2$). They come in three types (scalar, vector and tensor) which refers to the way they transform under a coordinate transformation on $\mathcal{S}^2$. As their name implies, scalar harmonics are invariant under these kinds of transformations, whereas vector harmonics change in accordance with the transformation rules of a vector and tensor harmonics in accordance with the transformation rules of a tensor. Every function on the sphere can be expressed as a sum of harmonics since each type forms a complete and orthonormal basis on $\mathcal{S}^2$.

The spherical harmonics can also be divided according to their parity. There are two parity modes: *even* and *odd*.

- Even-parity modes are modes that are symmetric under inversion, meaning

$$Y^{\ell m}(-\theta, \phi) = Y^{\ell m}(\theta, \phi). \tag{276}$$

  This is true if they transform as

$$Y^{\ell m}(\pi - \theta, \pi + \phi) = (-1)^{\ell} Y^{\ell m}(\theta, \phi) \tag{277}$$

  with ℓ an integer number.

- Odd-parity modes are those that are antisymmetric under inversion, so

$$X^{\ell m}(-\theta, \phi) = -X^{\ell m}(\theta, \phi). \tag{278}$$

  This is true if they transform as

$$X^{\ell m}(\pi - \theta, \pi + \phi) = (-1)^{\ell+1} X^{\ell m}(\theta, \phi). \tag{279}$$

Here ℓ and m are integers, with $\ell \geq 0$ and $-\ell \leq m \leq \ell$. It is well-known that the $\ell = 0$ and $\ell = 1$ mode are non-radiating in the context of black hole perturbations, and therefore $\ell \geq 2$ in this thesis.

The division of harmonics by their parity is of great convenience in the study of linear perturbations in a spherically symmetric background, where they are naturally prohibited from mixing. In this appendix we will explain the basic properties of spherical harmonics needed in our study of black hole perturbation theory. This is a summary of the relevant information from Martel's Appendix A [9].

B.1 Scalar spherical harmonics

Scalar spherical harmonics are the "usual", well-known functions $Y^{\ell m}$, and are defined by the eigenvalue equation

$$D^A D_A Y^{\ell m} = -\ell(\ell+1) Y^{\ell m}. \tag{280}$$

These harmonics can be used to decompose a scalar function $S(x^A)$ on $\mathcal{S}^2$ as

$$S(x^A) = \sum_{\ell,m} s_{\ell m} Y^{\ell m}(x^A). \tag{281}$$

The coefficients $s_{\ell m}$ can be found using the orthonormality relations for spherical harmonics (see [9]).

B.2 Vector spherical harmonics

Vector spherical harmonics come in two flavours; even parity, defined by

$$Y_A^{\ell m} := D_A Y^{\ell m}, \tag{282}$$

and odd parity, defined by

$$X_A^{\ell m} := -\varepsilon_A{}^B D_B Y^{\ell m}. \tag{283}$$

Here, ε_{AB} is the Levi-Civita tensor on $\mathcal{S}^2$. It is defined via the Levi-Civita symbol $\tilde{\varepsilon}_{AB}$ as [37]

$$\varepsilon_{AB} = \sqrt{|\det \Omega_{AB}|}\,\tilde{\varepsilon}_{AB} = \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix}, \tag{284}$$

such that the only non-zero components are $\varepsilon_{\theta\phi} = -\varepsilon_{\phi\theta} = \sin\theta$. Since ε_{AB} is totally anti-symmetric ($\varepsilon_{AB} = -\varepsilon_{BA}$), we have to be careful with the order of the indices in this tensor., as opposed to symmetric matrices like g_{ab} and Ω_{AB}. The Levi-Civita tensor obeys

$$D_C \varepsilon_{AB} = 0, \tag{285}$$

which can be seen by writing out each of the individual components.

Any vector function $V_A(x^B)$ can be decomposed in vector harmonics as

$$V_A(x^B) = \sum_{\ell,m} \left\{ v_{\ell m} Y_A^{\ell m}(x^B) + w_{\ell m} X_A^{\ell m}(x^B) \right\}. \tag{286}$$

The coefficients $v_{\ell m}$ and $w_{\ell m}$ are again determined by the orthogonality relations. Vector harmonics of even parity and odd parity are always orthogonal, since [21]

$$\int \bar{Y}^A_{\ell m} X_A^{\ell' m'} d\Omega = 0, \tag{287}$$

where the bar indicates complex conjugation and $d\Omega := \sin\theta d\theta d\phi$ is the volume element on $\mathcal{S}^2$.

B.3 Tensor spherical harmonics

To describe the perturbation component γ_{AB} in this thesis (a symmetric rank-2 tensor with three independent components), we need three rank-2 tensor spherical harmonics. We can describe the trace (scalar) part by $\Omega_{AB}Y^{\ell m}$, which has the transformation properties of even-parity modes. The non-trace part of γ_{AB} can be described by two distinct combinations. The even-parity combination is

$$Y^{\ell m}_{AB} := \left[D_A D_B + \frac{\ell(\ell+1)}{2}\Omega_{AB}\right] Y^{\ell m}, \tag{288}$$

while the odd-parity combination is

$$X^{\ell m}_{AB} := -\frac{1}{2}\left(\varepsilon_A{}^C D_B + \varepsilon_B{}^C D_A\right) D_C Y^{\ell m}. \tag{289}$$

Together they form a complete and orthogonal basis for symmetric rank-2 tensors. This means that generally one can decompose any symmetric tensor $T_{AB}(x^C)$ as

$$T_{AB}(x^C) = \sum_{\ell,m}\left\{V^{\ell m}\Omega_{AB}Y^{\ell m} + W^{\ell m}Y^{\ell m}_{AB} + U^{\ell m}X^{\ell m}_{AB}\right\}. \tag{290}$$

The functions $V^{\ell m}$, $W^{\ell m}$ and $U^{\ell m}$ again satisfy orthogonality relations. The harmonics themselves are orthogonal (but not orthonormal) with respect to each other:

$$\int \bar{Y}^{AB}_{\ell m} X^{\ell' m'}_{AB}\, d\Omega = 0, \tag{291}$$

$$\int \bar{Y}^{AB}_{\ell m} \Omega_{AB} Y^{\ell' m'}\, d\Omega = 0, \tag{292}$$

$$\int \bar{X}^{AB}_{\ell m} \Omega_{AB} Y^{\ell' m'}\, d\Omega = 0. \tag{293}$$

B.4 Identities

In this section we derive some properties that are needed in the derivation of the perturbation equations, providing short clarifying derivations. In these derivations we will use the square bracket above two covariant derivatives,

$$\overbracket{D_A D_B}, \tag{294}$$

to indicate that they are commuted in the next step. We use Eqs. (26)-(28) and the Riemann tensors, Eqs. (18) and (19), whenever we perform such a commutation.

The identities for $Y^{\ell m}_A$ are rather straightforward to prove, since $Y^{\ell m}$ is a scalar function. Therefore, the covariant derivatives working on $Y^{\ell m}$ may freely be commuted. We have

$$D_B Y^{\ell m}_A = D_B D_A Y^{\ell m} = D_A D_B Y^{\ell m} = D_A Y^{\ell m}_B, \tag{295}$$

which means that the tensor object $D_B Y_A^{\ell m}$ is symmetric. Eq. (280) furthermore implies that

$$
\begin{aligned}
D^A Y_A^{\ell m} &= D^A D_A Y^{\ell m} \\
&:= -\ell(\ell+1) Y^{\ell m}, && (296) \\
D_A D^B Y_B^{\ell m} &= D_A \left(-\ell(\ell+1) Y^{\ell m}\right) \\
&= -\ell(\ell+1) Y_A^{\ell m}, && (297) \\
D^B D_B Y_A^{\ell m} &= \overline{D^B D_A} Y_B^{\ell m} \\
&= D_A D^B Y_B^{\ell m} + R^B{}_{AB}{}^C Y_C^{\ell m} \\
&= -\ell(\ell+1) Y_A^{\ell m} + \left(\Omega^B_B \Omega^C_A - \Omega^{BC}\Omega_{AB}\right) Y_C^{\ell m} \\
&= \left[1 - \ell(\ell+1)\right] Y_A^{\ell m}. && (298)
\end{aligned}
$$

By a similar analysis for the odd-parity vector harmonics, we obtain

$$
\begin{aligned}
D^A X_A^{\ell m} &= -\varepsilon_A{}^B D^A D_B Y^{\ell m} = 0, && (299) \\
D^B D_B X_A^{\ell m} &= -\varepsilon_A{}^C D^B D_B Y_C^{\ell m} \\
&= -\varepsilon_A{}^C \left[1 - \ell(\ell+1)\right] Y_C^{\ell m} \\
&:= \left[1 - \ell(\ell+1)\right] X_A^{\ell m}, && (300) \\
D^B D_A X_B &= \Omega^{BC} D_C D_A \left(-\varepsilon_B{}^E D_E Y^{\ell m}\right) \\
&= -\varepsilon^{CE} \overline{D_C D_A} D_E Y^{\ell m} \\
&= -\varepsilon^{CE} \left(D_A D_C D_E + R_{CAE}{}^F D_F\right) Y^{\ell m} \\
&= -D_A \left(\cancel{\varepsilon^{CE} D_C D_E Y^{\ell m}}\right) - \varepsilon^{CE} \left(\Omega_{CE}\Omega^F_A - \Omega^F_C \Omega_{AE}\right) D_F Y^{\ell m} \\
&= -\cancel{\varepsilon^C{}_C} D_A Y^{\ell m} + \varepsilon^F{}_A D_F Y^{\ell m} \\
&= -\varepsilon_A{}^F D_F Y^{\ell m} \\
&= X_A^{\ell m}. && (301)
\end{aligned}
$$

Here we used that the derivative of the Levi-Civita tensor is zero. For the first and third identities we also used the anti-symmetry of the Levi-Civita tensor, which means for instance that contracting ε^{AB} with the symmetric tensor quantity $D_A D_B Y^{\ell m}$ gives zero.

For the tensor harmonics, one can show using Eq. (288), (289) and (280) that

$$
\Omega^{AB} Y_{AB}^{\ell m} = 0 = \Omega^{AB} X_{AB}^{\ell m}. \qquad (302)
$$

C Appendix: Covariant derivatives of (co-)vectors and (co-)tensors in the $\mathcal{M}^2 \times \mathcal{S}^2$ split

In this thesis we make use of covariant derivatives of (co-)vectors and (co-)tensors in terms of quantities on $\mathcal{M}^2$ and $\mathcal{S}^2$. The expressions below can be found in a different notation in [9], but for completeness we include them here in our notation.

We start with the first-order covariant derivatives. $\nabla_\alpha v^\beta$ has components [39]

$$\begin{aligned}
\nabla_a v^b &= \partial_a v^b + \Gamma^b_{ac} v^c = \mathscr{D}_a v^b, \\
\nabla_a v^B &= \partial_a v^B + \Gamma^B_{aA} v^A = \mathscr{D}_a v^B + \frac{1}{r} r_a v^B, \\
\nabla_A v^b &= \partial_A v^b + \Gamma^b_{AB} V^B = D_A v^b - r r^b \Omega_{AB} v^B, \\
\nabla_A v^B &= \partial_A v^B + \Gamma^B_{AC} v^C + \Gamma^B_{Ac} v^c = D_A v^B + \frac{1}{r} r_c \delta^B_A v^c.
\end{aligned} \tag{303}$$

We used here that $\mathscr{D}_a$ works on v^B as a scalar, as does D_A on v^b, such that we express the final results in terms of covariant derivatives instead of partial derivatives. Similarly for the co-vectors, the components of $\nabla_\alpha v_\beta$ are

$$\begin{aligned}
\nabla_a v_b &= \partial_a v_b - \Gamma^c_{ab} v_c = \mathscr{D}_a v_b, \\
\nabla_a v_B &= \partial_a v_B - \Gamma^A_{aB} v_A = \mathscr{D}_a v_B - \frac{1}{r} r_a v_B, \\
\nabla_A v_b &= \partial_A v_b - \Gamma^B_{Ab} v_B = D_A v_b - \frac{1}{r} r_b v_A, \\
\nabla_A v_B &= \partial_A v_B - \Gamma^C_{AB} v_C - \Gamma^c_{AB} v_c = D_A v_B + r r^c \Omega_{AB} v_c.
\end{aligned} \tag{304}$$

The same procedure also applies to higher derivatives and derivatives of higher-rank tensors. First-order covariant derivatives on a rank-2 tensor $t^{\mu\nu}$, are

$$\begin{aligned}
\nabla_a t^{bc} &= \mathscr{D}_a t^{bc}, \\
\nabla_a t^{bC} &= \mathscr{D}_a t^{bC} + \frac{1}{r} r_a t^{bC}, \\
\nabla_a t^{BC} &= \mathscr{D}_a t^{BC} + \frac{2}{r} r_a t^{BC}, \\
\nabla_A t^{bc} &= D_A t^{bc} - g^{ca} r t^{bB} \Omega_{AB} \mathscr{D}_a r - g^{ba} r r_a t^{Bc} \Omega_{AB}, \\
\nabla_A t^{bC} &= D_A t^{bC} + \frac{1}{r} r_a \delta_A{}^C t^{ba} - g^{ba} r r_a t^{BC} \Omega_{AB}, \\
\nabla_A t^{BC} &= D_A t^{BC} + \frac{1}{r} \delta_A{}^B t^{aC} \mathscr{D}_a r + \frac{1}{r} r_a \delta_A{}^C t^{Ba},
\end{aligned} \tag{305}$$

where $\delta_A{}^B$ is a Kronecker delta. On a co-tensor $t_{\mu\nu}$, we obtain

$$\begin{aligned}
\nabla_a t_{bc} &= \mathscr{D}_a t_{bc}, \\
\nabla_a t_{bC} &= \mathscr{D}_a t_{bC} - \frac{1}{r} r_a t_{bC}, \\
\nabla_a t_{BC} &= \mathscr{D}_a t_{BC} - \frac{2}{r} r_a t_{BC}, \\
\nabla_A t_{bc} &= D_A t_{bc} - \frac{1}{r} r_b t_{Ac} - \frac{1}{r} r_c t_{bA}, \\
\nabla_A t_{bC} &= D_A t_{bC} - \frac{1}{r} r_b t_{AC} + r r^a \Omega_{AC} t_{ba}, \\
\nabla_A t_{BC} &= D_A t_{BC} + r r^a \Omega_{AB} t_{aC} + r r^a \Omega_{AC} t_{Ba}.
\end{aligned} \tag{306}$$

The first-order derivatives are not needed in the calculations of the perturbation equations, but are used to calculate second-order derivatives of rank-2 tensors. On a tensor with two lowercase Latin indices we obtain

$$\begin{aligned}
\nabla_c\nabla_d t_{ab} &= \mathscr{D}_c\mathscr{D}_d t_{ab}, \\
\nabla_A\nabla_c t_{ab} &= D_A\mathscr{D}_c t_{ab} - \frac{r_c}{r}\left(D_A t_{ab} - \frac{4}{r}r_{(a}t_{b)A}\right) - \frac{2}{r}r_{(a}\mathscr{D}_{|c|}t_{b)A}, \\
\nabla_c\nabla_A t_{ab} &= \nabla_A\nabla_c t_{ab} - \frac{2}{r}(\mathscr{D}_{(a}r_{|c|})t_{b)A}, \\
\nabla_B\nabla_A t_{ab} &= D_B D_A t_{ab} - \frac{2}{r}r_{(a}D_B t_{b)A} - \frac{2}{r}r_{(a}D_A t_{b)B} + \frac{2}{r^2}r_a r_b t_{AB} \\
&\quad + r r^c\Omega_{AB}\left(\mathscr{D}_c t_{ab} - \frac{2}{r}r_{(a}t_{b)c}\right),
\end{aligned} \tag{307}$$

and on a tensor with one lowercase Latin index and one capital index we find

$$\begin{aligned}
\nabla_c\nabla_b t_{aA} &= \mathscr{D}_c\mathscr{D}_b t_{aA} - \frac{2}{r}r_{(c}\mathscr{D}_{b)}t_{aA} - \frac{1}{r}\left(\mathscr{D}_c r_b - \frac{2}{r}r_b r_c\right)t_{aA}, \\
\nabla_B\nabla_b t_{aA} &= D_B\mathscr{D}_b t_{aA} - \frac{2}{r}r_b D_B t_{aA} - \frac{r_a}{r}\left(\mathscr{D}_b t_{AB} - \frac{3}{r}r_b t_{AB}\right) + r r^c\Omega_{AB}\left(\mathscr{D}_b t_{ac} - \frac{r_b}{r}t_{ac}\right), \\
\nabla_b\nabla_B t_{aA} &= D_B\mathscr{D}_b t_{aA} - \frac{1}{r}(\mathscr{D}_b r_a)t_{AB} + r\Omega_{AB}(\mathscr{D}_b r^c)t_{ac}, \\
\nabla_C\nabla_B t_{aA} &= D_C D_B t_{aA} - \frac{2}{r}r_a D_{(C}t_{|A|B)} + r r^b\left(\Omega_{BC}\mathscr{D}_b t_{aA} - \frac{r_a}{r}\Omega_{AB}t_{bC}\right) \\
&\quad + r r^b\left(2\Omega_{A(C}D_{B)}t_{ab} - \frac{2}{r}\Omega_{BC}r_{(a}t_{b)A} - \frac{2}{r}\Omega_{AC}r_{(a}t_{b)B}\right).
\end{aligned} \tag{308}$$

Finally, on two lowercase capital indices we obtain

$$\begin{aligned}
\nabla_b\nabla_a t_{AB} &= \mathscr{D}_b\mathscr{D}_a t_{AB} - \frac{4}{r}r_{(b}\mathscr{D}_{a)}t_{AB} + \frac{6}{r^2}r_a r_b t_{AB} - \frac{2}{r}\mathscr{D}_b r_a t_{AB}, \\
\nabla_C\nabla_a t_{AB} &= D_C\mathscr{D}_a t_{AB} - \frac{3}{r}r_a D_C t_{AB} + 2r r^b\Omega_{C(A}\left(\mathscr{D}_a t_{B)b} - \frac{2}{r}r_a t_{B)b}\right), \\
\nabla_a\nabla_C t_{AB} &= D_C\mathscr{D}_a t_{AB} + 2r(\mathscr{D}_a r^b)\Omega_{C(A}t_{B)b}, \\
\nabla_D\nabla_C t_{AB} &= D_D D_C t_{AB} + 2r r^a(\Omega_{C(A}D_D t_{B)a} + \Omega_{D(A}D_C t_{B)a}) - 2r^a r_a(\Omega_{D(A}t_{B)C} + \Omega_{CD}t_{AB}) \\
&\quad + 2r^2 r^a r^b\Omega_{D(A}\Omega_{B)C}t_{ab} + r r^a\Omega_{CD}\mathscr{D}_a t_{AB}.
\end{aligned} \tag{309}$$

Note that the round parentheses indicate symmetrization and $|\cdot|$ means the index $\cdot$ is left out of the symmetrization process.

D Appendix: Explicit calculation of Linearized Ricci and Einstein tensors

In this appendix, we will derive the covariant components of the linearized Ricci tensor and Einstein tensor in terms of the dimension $d = 2, 3$, where $d = 2$ corresponds to the Schwarzschild case, and $d = 3$ to the black string. We make use of the fact that the odd- and even-parity sectors of the spherical harmonic decomposition, Eqs. (84) and (105), are the same in four and five dimensions in the RW gauge (in four dimensions, the RW gauge allows us to remove the modes h_2, j_a and G, while in five dimensions we can remove h_3, j_a and G).

Starting with the odd-parity perturbation equations, we insert the spherical harmonics expansion of Eqs. (84) into Eq. (52), which gives

$$
\begin{aligned}
\delta R^{(\mathrm{odd})}_{ab} &= \mathscr{D}_m \cancel{C^{m}_{ab}} + \frac{2}{r} r_m \cancel{C^{m}_{ab}} - \frac{1}{2}\mathscr{D}_a\mathscr{D}_b \cancel{\gamma^{m}_{m}} - \frac{1}{2r^2} D^M D_M \cancel{\gamma_{ab}} + \frac{1}{2r^2} D_M \left(\mathscr{D}_a \gamma^M_b + \mathscr{D}_b \gamma^M_a\right) \\
&\quad - \frac{1}{2r^2}\mathscr{D}_a\mathscr{D}_b \cancel{\gamma^{M}_{M}} + \frac{1}{2r^3}\left(r_a \mathscr{D}_b \cancel{\gamma^{M}_{M}} + r_b \mathscr{D}_a \cancel{\gamma^{M}_{M}}\right) - \frac{1}{r^4}\left(r_a r_b - r\mathscr{D}_a r_b\right) \cancel{\gamma^{M}_{M}} \\
&= \frac{1}{2r^2} D_M \left(\mathscr{D}_a \gamma^M_b + \mathscr{D}_b \gamma^M_a\right) \\
&= \frac{1}{2r^2}\left(\mathscr{D}_a h_b + \mathscr{D}_b h_a\right) \cancel{D^M X_M} \\
&= 0.
\end{aligned} \tag{310}
$$

In the first step, many of the terms were directly zero by virtue of Eq. (84). The last step gives zero because of property (299). Similarly, Eq. (53) becomes

$$
\begin{aligned}
\delta R^{(\mathrm{odd})}_{aB} &= \frac{1}{2} D_B \left(\mathscr{D}_m \cancel{\gamma^{m}_{a}} - \mathscr{D}_a \cancel{\gamma^{m}_{m}} + \frac{1}{r} r_a \cancel{\gamma^{m}_{m}}\right) - \frac{1}{2}\left(\overset{(\wedge)}{\Box}\gamma_{aB} - \mathscr{D}_m \mathscr{D}_a \gamma^m_B\right) - \frac{1}{r}\left(r_a \mathscr{D}_m \gamma^m_B - r_m \mathscr{D}_a \gamma^m_B\right) \\
&\quad - \frac{1}{r^2}\left(r_a r_m + r\mathscr{D}_a r_m\right)\gamma^m_B + \frac{1}{2r^2} D^M \left(D_B \gamma_{aM} - D_M \gamma_{aB}\right) + \frac{1}{2r^2}\mathscr{D}_a \left(D_M \cancel{\gamma^{M}_{B}} - D_B \cancel{\gamma^{M}_{M}}\right) \\
&\quad - \frac{1}{r^3} r_a \left(D_M \cancel{\gamma^{M}_{B}} - D_B \cancel{\gamma^{M}_{M}}\right) \\
&= \left[-\frac{1}{2}\Box h_a + \frac{1}{2}\overbrace{\mathscr{D}^b \mathscr{D}_a} h_b - \frac{1}{r} r_a \mathscr{D}^b h_b + \frac{1}{r} r^b \mathscr{D}_a h_b - \frac{1}{r^2} r_a r^b h_b - \frac{1}{r} h_b \mathscr{D}_a \mathscr{D}^b r\right] X_B \\
&\quad + \left[\frac{1}{2r^2} h_a D^A D_B\right] X_A - \left[\frac{1}{2r^2} h_a D^A D_A\right] X_B \\
&= \Bigg[-\frac{1}{2}\overset{(\wedge)}{\Box} h_a + \frac{1}{2}\mathscr{D}_a \mathscr{D}^b h_b - \frac{1}{r} r_a \mathscr{D}^b h_b + \frac{1}{r} r^b \mathscr{D}_a h_b - \frac{1}{r^2} r_a r^b h_b \\
&\quad - \frac{1}{r} h_b \mathscr{D}_a \mathscr{D}^b r + \left(\frac{\ell(\ell+1)}{2r^2} + \frac{2M}{r^3 d}\right) h_a\Bigg] X_B.
\end{aligned} \tag{311}
$$

In the last step, we commuted the covariant derivatives in the second term as

$$
\begin{aligned}
\mathscr{D}^b \mathscr{D}_a h_b &= g^{bc} \mathscr{D}_c \mathscr{D}_a h_b \\
&= g^{bc}\left(\mathscr{D}_a \mathscr{D}_c h_b + R_{cabd} h^d\right) \\
&= \mathscr{D}_a \mathscr{D}^b h_b + \frac{4M}{r^3 d(d-1)} g^{bc}\left(g_{cb} g_{ad} - g_{cd} g_{ab}\right) h^d \\
&= \mathscr{D}_a \mathscr{D}^b h_b + \frac{4M}{r^3 d} h_a,
\end{aligned}
$$

and for the last two terms we used properties (300) and (301) from Appendix A. The d'Alembertian operator $\overset{(\wedge)}{\Box}$ is to be identified as $\Box$ if $d = 2$ and $\hat{\Box}$ if $d = 3$.

Finally, Eq. (54) becomes

$$\begin{aligned}
\delta R^{(\mathrm{odd})}_{AB} &= \Omega_{AB}\left[r r_a \mathscr{D}_b \left(\cancel{\gamma^{ab}} - \frac{1}{2} g^{ab} \cancel{\gamma^m_m} \right) + \left(r_a r_b + r \mathscr{D}_a r_b \right) \cancel{\gamma^{ab}} \right] - \frac{1}{2} D_A D_B \cancel{\gamma^a_a} \\
&\quad + \frac{1}{2} \mathscr{D}_a \left(D_A \gamma^a_B + D_B \gamma^a_A \right) + \frac{1}{r} r_a \Omega_{AB} D_M \gamma^{aM} - \frac{1}{2} \overset{(\wedge)}{\Box} \cancel{\gamma_{AB}} + \frac{1}{r^2} D_M \cancel{C^M_{AB}} - \frac{1}{2r^2} D_A D_B \cancel{\gamma^M_M} \\
&\quad + \frac{1}{r} r^a \mathscr{D}_a \left(\cancel{\gamma_{AB}} - \frac{1}{2} \Omega_{AB} \cancel{\gamma^M_M} \right) - \frac{2}{r^2} r^a r_a \left(\cancel{\gamma_{AB}} - \frac{1}{2} \Omega_{AB} \cancel{\gamma^M_M} \right) \\
&= \frac{1}{2} \mathscr{D}_a \left(D_A \gamma^a_B + D_B \gamma^a_A \right) + \frac{1}{r} r_a \Omega_{AB} D_M \gamma^{aM} \\
&= \frac{1}{2} \mathscr{D}^a h_a \left(D_A X_B + D_B X_A \right) + \frac{1}{r} r^a \Omega_{AB} h_a \cancel{D^M X_M} \\
&= -\frac{1}{2} \mathscr{D}^a h_a \left(\varepsilon^F_B D_A + \varepsilon^F_A D_B \right) D_F Y \\
&:= [\mathscr{D}^a h_a] X_{AB}.
\end{aligned} \tag{312}$$

We continue with the even-parity case. Inserting Eqs. (105) into Eq. (52), we obtain

$$\begin{aligned}
\delta R^{(\mathrm{even})}_{ab} &= \mathscr{D}_m C^m_{ab} + \frac{2}{r} r_m C^m_{ab} - \frac{1}{2} \mathscr{D}_a \mathscr{D}_b \gamma^m_m - \frac{1}{2r^2} D^M D_M \gamma_{ab} + \frac{1}{2r^2} D_M \left(\mathscr{D}_a \cancel{\gamma^M_b} + \mathscr{D}_b \cancel{\gamma^M_a} \right) \\
&\quad - \frac{1}{2r^2} \mathscr{D}_a \mathscr{D}_b \gamma^M_M + \frac{1}{2r^3} \left(r_a \mathscr{D}_b \gamma^M_M + r_b \mathscr{D}_a \gamma^M_M \right) - \frac{1}{r^4} \left(r_a r_b - r \mathscr{D}_a r_b \right) \gamma^M_M \\
&= \frac{1}{2} \mathscr{D}_m \left(\mathscr{D}_b f^m_a + \mathscr{D}_a f^m_b - \mathscr{D}^m f_{ab} \right) Y + \frac{1}{r} r_m \left(\mathscr{D}_b f^m_a + \mathscr{D}_a f^m_b - \mathscr{D}^m f_{ab} \right) Y \\
&\quad - \frac{1}{2} \mathscr{D}_a \mathscr{D}_b f^m_m Y - \frac{1}{2r^2} f_{ab} D^M D_M Y - \frac{1}{2r^2} \mathscr{D}_a \mathscr{D}_b \left(r^2 \Omega^M_M K \right) Y \\
&\quad + \frac{1}{2r^3} \left(r_a \mathscr{D}_b \left(r^2 \Omega^M_M K \right) Y + r_b \mathscr{D}_a \left(r^2 \Omega^M_M K \right) Y \right) - \frac{1}{r^4} \left(r_a r_b - r \mathscr{D}_a r_b \right) \left(r^2 \Omega^M_M K \right) Y \\
&= \left[\frac{1}{2} \mathscr{D}_m \left(\mathscr{D}_b f^m_a + \mathscr{D}_a f^m_b - \mathscr{D}^m f_{ab} \right) + \frac{1}{r} r_m \left(\mathscr{D}_b f^m_a + \mathscr{D}_a f^m_b - \mathscr{D}^m f_{ab} \right) - \frac{1}{2} \mathscr{D}_a \mathscr{D}_b f^m_m \right. \\
&\quad + \frac{\ell(\ell+1)}{2r^2} f_{ab} - \frac{1}{r^2} \mathscr{D}_a \mathscr{D}_b \left(r^2 K \right) + \frac{1}{r^3} \left(r_a \mathscr{D}_b \left(r^2 K \right) + r_b \mathscr{D}_a \left(r^2 K \right) \right) \\
&\quad \left. - \frac{2}{r^2} \left(r_a r_b - r \mathscr{D}_a r_b \right) K \right] Y \\
&= \frac{1}{2} \left[\mathscr{D}_b \mathscr{D}_m f^m_a + \frac{8M}{r^3 d(d-1)} \left(d f_{ba} - g_{ba} f^m_m \right) + \mathscr{D}_a \mathscr{D}_m f^m_b - \overset{(\wedge)}{\Box} f_{ab} \right. \\
&\quad + \frac{2}{r} r_m \left(\mathscr{D}_b f^m_a + \mathscr{D}_a f^m_b - \mathscr{D}^m f_{ab} \right) - \mathscr{D}_a \mathscr{D}_b f^m_m + \frac{\ell(\ell+1)}{r^2} f_{ab} \\
&\quad \left. - \frac{2}{r} \left(r_b \mathscr{D}_a K - r_a \mathscr{D}_b K \right) - 2 \mathscr{D}_a \mathscr{D}_b K \right] Y.
\end{aligned} \tag{313}$$

In the third step, the covariant derivatives in the first term were commuted according to Eq. (26).

We also have

$$\begin{aligned}
\delta R^{(\text{even})}_{aB} &= \frac{1}{2} D_B \left(\mathscr{D}_b \gamma^b_a - \mathscr{D}_a \gamma^b_b + \frac{1}{r} r_a \gamma^b_b \right) - \frac{1}{2} \left(\overset{(\wedge)}{\Box} \cancel{\gamma_{aB}} - \mathscr{D}_b \mathscr{D}_a \cancel{\gamma^b_B} \right) - \frac{1}{r} \left(r_a \mathscr{D}_b \cancel{\gamma^b_B} - r_b \mathscr{D}_a \cancel{\gamma^b_B} \right) \\
&\quad - \frac{1}{r^2} \left(r_a r_b + r \mathscr{D}_a r_b \right) \cancel{\gamma^b_B} + \frac{1}{2r^2} D^M \left(D_B \cancel{\gamma_{aM}} - D_M \cancel{\gamma_{aB}} \right) - \frac{1}{2r^2} \mathscr{D}_a \left(D_M \gamma^M_B - D_B \gamma^M_M \right) \\
&\quad - \frac{1}{r^3} r_a \left(D_M \gamma^M_B - D_B \gamma^M_M \right) \\
&= \frac{1}{2} D_B \left(\mathscr{D}_b f^b_a - \mathscr{D}_a f^b_b + \frac{1}{r} r_a f^b_b \right) Y + \frac{1}{2r^2} \mathscr{D}_a \left(D_M (r^2 K \Omega^M_B Y) - D_B (r^2 K \Omega^M_M Y) \right) \\
&\quad - \frac{1}{r^3} r_a \left(D_M (r^2 K \Omega^M_B Y) - D_B (r^2 K \Omega^M_M Y) \right) \\
&= \frac{1}{2} D_B \left(\mathscr{D}_b f^b_a - \mathscr{D}_a f^b_b + \frac{1}{r} r_a f^b_b \right) D_B Y - \frac{1}{2r^2} \mathscr{D}_a (r^2 K) D_B Y + \frac{1}{r} r_a K D_B Y \\
&= \frac{1}{2} \left[\mathscr{D}_b f^b_a - \mathscr{D}_a f^b_b + \frac{1}{r} r_a f^b_b - \mathscr{D}_a K \right] Y_B. \qquad (314)
\end{aligned}$$

Lastly,

$$\begin{aligned}
\delta R^{(\text{even})}_{AB} &= \Omega_{AB} \left[r r_a \mathscr{D}_b \left(\gamma^{ab} - \frac{1}{2} g^{ab} \gamma^m_m \right) + \left(r_a r_b + r \mathscr{D}_a r_b \right) \gamma^{ab} \right] - \frac{1}{2} D_A D_B \gamma^a_a \\
&\quad + \frac{1}{2} \mathscr{D}_a \left(D_A \cancel{\gamma^a_B} + D_B \cancel{\gamma^a_A} \right) + \frac{1}{r} r_a \Omega_{AB} D_M \cancel{\gamma^{aM}} - \frac{1}{2} \overset{(\wedge)}{\Box} \gamma_{AB} + \frac{1}{r^2} D_M C^M_{AB} - \frac{1}{2r^2} D_A D_B \gamma^M_M \\
&\quad + \frac{1}{r} r^a \mathscr{D}_a \left(\gamma_{AB} - \frac{1}{2} \Omega_{AB} \gamma^M_M \right) - \frac{2}{r^2} r^a r_a \left(\gamma_{AB} - \frac{1}{2} \Omega_{AB} \gamma^M_M \right) \\
&= \Omega_{AB} \left[r r_a \mathscr{D}_b \left(f^{ab} - \frac{1}{2} g^{ab} f^m_m \right) + \left(r_a r_b + r \mathscr{D}_a r_b \right) f^{ab} \right] Y - \frac{1}{2} f^a_a D_A D_B Y - \frac{1}{2} \overset{(\wedge)}{\Box} (r^2 K) \Omega_{AB} Y \\
&\quad + \frac{1}{2r^2} D_M \left[D_B (r^2 K \Omega^M_A Y) + D_A (r^2 K \Omega^M_B Y) - D^M (r^2 K \Omega_{AB} Y) \right] - \frac{1}{2r^2} D_A D_B (r^2 K \Omega^M_M Y) \\
&\quad + \frac{1}{r} r^a \mathscr{D}_a (r^2 K \Omega_{AB} Y - \frac{1}{2} \Omega_{AB} r^2 K \Omega^M_M Y) - \frac{2}{r^2} r^a r_a (r^2 K \Omega_{AB} Y - \frac{1}{2} \Omega_{AB} r^2 K \Omega^M_M Y) \\
&= \Omega_{AB} \left[r r_a \mathscr{D}_b \left(f^{ab} - \frac{1}{2} g^{ab} f^m_m \right) + \left(r_a r_b + r \mathscr{D}_a r_b \right) f^{ab} \right] Y - \frac{1}{2} f^a_a D_A D_B Y - \frac{1}{2} \overset{(\wedge)}{\Box} (r^2 K) \Omega_{AB} Y \\
&\quad + \frac{1}{2} K (D_A D_B Y + D_B D_A Y - \Omega_{AB} D^M D_M Y) - K D_A D_B Y \qquad (315) \\
&= \left[r r_a \mathscr{D}_b f^{ab} - \frac{1}{2} r r^b \mathscr{D}_b f^m_m + r_a r_b f^{ab} + r \mathscr{D}_a r_b f^{ab} - \frac{1}{2} \overset{(\wedge)}{\Box} (r^2 K) + \frac{1}{2} \ell(\ell+1) K \right. \\
&\quad \left. + \frac{1}{4} \ell(\ell+1) f^a_a \right] \Omega_{AB} Y - \frac{1}{2} f^a_a Y_{AB}. \qquad (316)
\end{aligned}$$

The Einstein tensors are calculated in a similar manner but require significantly more work. For these we use Eqs. (55)-(57) and expand them in spherical harmonics. The multipole expansions for each individual term can be found in Appendix B.2 of Martel [9], which can be translated to our notation using the notation conversion in Table 4. We merely state the final results, of which we only need the even-parity part in this thesis:

$$\begin{aligned}
\delta G_{ab}^{(\text{even})} &= \Bigg[\mathscr{D}_c\mathscr{D}_{(b}f_{a)}^c - \frac{1}{2}g_{ab}\mathscr{D}_c\mathscr{D}_d f^{cd} - \frac{1}{2}\mathscr{D}_a\mathscr{D}_b f_c^c - \frac{1}{2}(\overset{(\wedge)}{\Box} f_{ab} - g_{ab}\overset{(\wedge)}{\Box} f_c^c) \\
&\quad + \frac{2}{r}r_c(\mathscr{D}_{(b}f_{a)}^c - g_{ab}\mathscr{D}_d f^{cd}) - \frac{1}{r}r^c(\mathscr{D}_c f_{ab} - g_{ab}\mathscr{D}_c f_d^d) + \frac{\ell(\ell+1)}{2r^2}f_{ab} \\
&\quad - \frac{1}{2}g_{ab}\left(\frac{2}{r^2}r^c r^d f_{cd} + \frac{2}{r}(\mathscr{D}_c r_d)f^{cd} + \frac{\ell(\ell+1)}{r^2}f_c^c\right) - \mathscr{D}_a\mathscr{D}_b K + g_{ab}\overset{(\wedge)}{\Box} K \\
&\quad - \frac{2}{r}r_{(a}\mathscr{D}_{b)}K + \frac{3}{r}g_{ab}r^c\mathscr{D}_c K - \frac{1}{2}g_{ab}\left(-\frac{2}{r^2}(r\overset{(\wedge)}{\Box} r + r^c r_c) + \frac{\ell(\ell+1)}{r^2}K\right)\Bigg]Y, \qquad (317)\\
\delta G_{aB}^{(\text{even})} &= \frac{1}{2}\left[\mathscr{D}_b f_a^b - \mathscr{D}_a f_b^b + \frac{1}{r}r^a f_b^b - \mathscr{D}_a K\right]Y_B \qquad (318)\\
\delta G_{AB}^{(\text{even})} &= \frac{1}{2}r^2\left[\overset{(\wedge)}{\Box} f_a^a - \mathscr{D}_a\mathscr{D}_b f^{ab} - \frac{2}{r}r^b\mathscr{D}_a f_b^a + \frac{1}{r}r^a\mathscr{D}_a f_b^b - \frac{\ell(\ell+1)}{2r^2}f_a^a + \frac{2}{r}r^a\mathscr{D}_a K + \overset{(\wedge)}{\Box} K\right]\Omega_{AB}Y \\
&\quad - \frac{1}{2}f_a^a Y_{AB}. \qquad (319)
\end{aligned}$$

Clearly, (317)-(319) are equal[27] in four and five dimensions up to the dimensionality of the d'Alembertian operator (one can check that all identities in Martel's Appendix B.1 and B.2 hold in both four and five dimensions since they are derived using only the Christoffel symbols, which are identical in four and five dimensions). The even- parity part of Eqs. (317)-(319) can be identified simply by the parity of the harmonics.

[27] Note that e.g. f_{ab} does change in five dimensions compared to four, but this effect is only evident when we actually insert the coordinates, revealing the new variables H_3, H_4 and H_5.

Addendum

In the final stages of writing this thesis, we have noticed that it was a fundamental mistake to assume that $\mathcal{M}^3$ has constant curvature, and thus the validity of Eq. (195). Working out explicitly the left- and right-hand-side of (195) shows that they are not equal (up to a factor of 3). Therefore, the addition of a uniform spatial dimension does in fact induce non-constant curvature.

The implications of this mistake are that some of the calculations of Section 3 are incorrect. When commuting covariant derivatives, we were not allowed to use Eq. (195) for the Riemann tensor. Investigating the components of $\mathscr{R}_{abcd}$ in four and five dimensions shows that the non-zero components are unaffected. We therefore expect that the main results of Section 3 only differ in certain prefactors. For example, the odd-parity master equations (245) will likely be affected only in the numbers appearing in the potentials, while the overall structure remains unaffected. The results are therefore still useful in the sense that they prove that the odd-parity sector can be decoupled into two independent equations, although their exact form should be slightly corrected.